Probability : Limit Theorem II

Final Take - Home

I have not given or received any unauthorized assistance on this exam.

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#1. Given $\begin{cases} dX_\varepsilon(t) = X_\varepsilon(t)dt + \varepsilon dW(t) & , t \geq 0, \quad \varepsilon > 0. \\ X_\varepsilon(0) = 0 \end{cases}$

ⓐ. WTS: $X_\varepsilon(t) = \varepsilon e^{t} N(t)$, $N(t) := \int_0^t e^{-s} dW(s)$, $t \geq 0$.

Proof: Let $Z_\varepsilon(t) = e^{t} X_\varepsilon(t)$, then by Itô's Rule for $Z_\varepsilon(t) = g(t, X_\varepsilon(t))$, where $g : [0,+\infty) \times \mathbb{R} \to \mathbb{R} \in C^{1,2}$ is defined by $g(t,x) = e^{t} x$; we have that:

$$dz_\varepsilon(t) = \bar{e}^{t} X_\varepsilon(t) dt + \bar{e}^{t} dX_\varepsilon(t) \qquad \text{as } \frac{\partial^2}{\partial x^2} g \equiv 0.$$

$$= -e^{-t} X_\varepsilon(t) dt + \bar{e}^{t} X_\varepsilon(t) dt + \varepsilon e^{-t} dW(t). \quad \text{by assumption that } X_\varepsilon(t) \text{ is a solution.}$$

$$\Rightarrow Z_\varepsilon(t) = Z_\varepsilon(0) + \int_0^t \varepsilon e^{-s} dW(s)$$

$$= 0 + \varepsilon N(t) \qquad \text{by definition of } Z_\varepsilon(t) \text{ \& } N(t).$$

$$\bar{e}^{t} Z_\varepsilon(t) = X_\varepsilon(t) = \varepsilon e^{-t} N(t). \quad ; \forall t \geq 0, \text{ for each } \varepsilon > 0.$$

It remains to show that $X_\varepsilon(t)$ is a strong solution, and the equation satisfies the condition for uniqueness of strong solution.

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(i) $X_\varepsilon(t) = \varepsilon e^{t}\int_0^t e^{-s}\,dW_s$ is adapted to $\mathcal{F}_t^W$ as $N(t)$ is from construction

(ii) $P\{X_\varepsilon(0)=0\} = 1$ as $X_\varepsilon(0) = e^{t}\varepsilon\cdot 0 = 0$ everywhere.

(iii) Since $\langle X_\varepsilon\rangle_t = \varepsilon^2 e^{2t}\int_0^t e^{-2s}\,ds = \frac{\varepsilon^2}{2}\left(e^{2t}-1\right)$

for any fixed $\varepsilon \geq 0$, $t \geq 0$, $\exists\, B_t^\varepsilon < \infty$ s.t. $\langle X_\varepsilon\rangle_t < B_t^\varepsilon$, then

for $\forall\ \hat{\varepsilon} \geq 0$, we have $P\{\sup_{s\in[0,t]}|X_s| \leq a_{\hat{\varepsilon}}\} \geq 1-2e^{-\frac{a_{\hat{\varepsilon}}^2}{2B_t^\varepsilon}} \geq 1-\hat{\varepsilon}$

by exponential inequality.

$\Rightarrow$ $P\{\sup_{s\in[0,T]}|X_s| < +\infty\} = 1$ by letting $a_{\hat{\varepsilon}} \to +\infty$.

$\Rightarrow$ $P\{\int_0^t |X_s|\,ds + \int_0^t \varepsilon\,ds \leq +\infty\} = 1$ as $\begin{cases} \int_0^t |X_s|\,ds \leq t\cdot\sup_{s\in[0,t]}|X_s| \\ \int_0^t \varepsilon\,ds = t\varepsilon \end{cases}$

(iv) $X_\varepsilon(t)$ satisfies $dX_\varepsilon(t) = X_\varepsilon(t)\,dt + \varepsilon\,dW_t$ by Itô (by construction actually)

By (i), (ii), (iii), (iv), $X_\varepsilon(t)$ is a strong solution.

Since $|b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| = |x-y| \leq |x-y|$,

the S.D.E. satisfies global Lipschitz (hence local Lipschitz),

we have $X_\varepsilon(t)$ is the unique strong solution.

#1. (b). WTS: $N(t)$ converges a.s. to some R.V. Z as $t \to \infty$, and $Z \sim (0, \frac{1}{2})$

Proof: $N(t) := \int_0^t e^{-s} dW(s)$ by definition.

Pick a monotone increasing sequence $\{t_i\}_{i \in \mathbb{N}} \subset [0, \infty)$, and construct the sequence of R.V.'s $\{N(t_i)\}_{i \in \mathbb{N}}$.

Since for each $t \in [0, +\infty)$, e^{-s} is bdd on $[0, t]$, by HW#3, Q#17., we have $N(t) \sim \mathcal{N}(0, \int_0^t e^{-2s} ds = \frac{1}{2}(1 - e^{-2t}))$, $\forall t \in [0, +\infty)$

Note: $E|N(t_i) - N(t_j)|^2 = E\left(\int_{t_i}^{t_j} e^{-s} dW_s\right)^2$ assume $t_i < t_j$ WOLG.

$$= \int_{t_i}^{t_j} e^{-2s} ds \quad \text{as } N(t_i) - N(t_j) \in \mathcal{M}_2^c$$

$$= -\frac{1}{2}(e^{-2t_j} - e^{-2t_i}) = \frac{1}{2} e^{-2t_i}(1 - e^{-2(t_j - t_i)})$$

$$\to 0 \quad \text{as } i, j \to \infty$$

$\Rightarrow \{N(t_i)\}_{i \in \mathbb{N}}$ is a Cauchy sequence in L^2 (and L^2 is Banach)

$\Rightarrow \exists Z$ s.t. $N(t_i) \xrightarrow{L^2} Z$ as $i \to \infty$

Now, since $N(t_i) \sim \mathcal{N}(0, \frac{1}{2}(1 - e^{-2t_i}))$, and $N(t_i) \xrightarrow[i \to \infty]{L^2} Z$,

we have that Z is the P-a.s. limit of $\{N(t_{i_k})\}_{k \in \mathbb{N}}$

as L^2 convergence implies $\exists \{t_{i_k}\}_{k \in \mathbb{N}}$ s.t. $N(t_{i_k}) \xrightarrow[k \to +\infty]{} Z$ P-a.s.

Finally, since our choice of monotone $\{t_i\}_{i \in \mathbb{N}}$ is arbitrary, and the P-a.s. limit Z is independent of $\{t_i\}_{i \in \mathbb{N}}$, we have

$\lim_{t \to +\infty} N(t) = Z$ P-a.s.

$\square$

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Since by monotonicity, the L^2 limit and the P-a.s. limit agrees,
we have $N(t_i) \xrightarrow[i \to +\infty]{L^2} Z$, for $\{t_i\}_{i \in \mathbb{N}}$.

Since $N(t_i) \sim \mathcal{N}(0, \frac{1}{2}(1-e^{-2t_i}))$, and the L^2 limit of sequence of Gaussian R.V.'s is still Gaussian w/ limit Variance, we have:

$$N(t_i) \xrightarrow{L^2} Z \Rightarrow Z \sim \mathcal{N}(0, \lim_{t_i \to +\infty} \tfrac{1}{2}(1-e^{-2t_i}))$$

$$\Rightarrow Z \sim \mathcal{N}(0, \tfrac{1}{2})$$

#1. ©: Define $\tau_\varepsilon := \inf\{t \geq 0 : |X_\varepsilon(t)| = 1\}$.

WTS: ① $\tau_\varepsilon \to \infty$ a.s. as $\varepsilon \to 0$, and ② find the limiting dist. of $\tau_\varepsilon - \ln\frac{1}{\varepsilon}$

① Proof: for any $K \in \mathbb{N}$, we have $T \in [0, +\infty)$.

$$\{\tau_\varepsilon > \overset{T}{K}\} = \{\sup_{t\in[0,T]} |X_\varepsilon(t)| < 1\} = \quad \text{by definition}$$

$$= \{\sup_{t\in[0,T]} e^{t} N(t) < \tfrac{1}{\varepsilon}\} \quad \text{by definition}$$

$$\supset \{\sup_{t\in[0,T]} N(t) \leq \tfrac{1}{\varepsilon} e^{-T}\} \quad \text{as } e^{-t} \geq e^{-T}, \forall t \in [0,T]$$

$$\Rightarrow \{\lim_{\varepsilon\to 0^+} \tau_\varepsilon > K\} \supset \{\sup_{t\in[0,T]} N(t) \leq \infty\}$$

$$\Rightarrow P\{\lim_{\varepsilon\to 0^+} \tau_\varepsilon \geq K\} \geq P\{\sup_{t\in[0,T]} N(t) \leq \infty\} = 1 \quad , \forall K \in [0, +\infty).$$

Therefore, we showed that $\tau_\varepsilon \to \infty$ a.s. as $\varepsilon \to 0^+$.

□

②. $$\{\tau_\varepsilon - \ln\tfrac{1}{\varepsilon} \leq T\} = \{\sup_{t\in[\ln\frac{1}{\varepsilon}, T+\ln\frac{1}{\varepsilon}]} |X_\varepsilon(t)| \geq 1\}$$

$$= \{\sup_{t\in[\ln\frac{1}{\varepsilon}, T+\ln\frac{1}{\varepsilon}]} |N(t)| e^{t} \geq \tfrac{1}{\varepsilon}\}$$

$$\tilde{t} = t - \ln\tfrac{1}{\varepsilon} \Rightarrow = \{\sup_{\tilde{t}\in[0,T]} N(\tilde{t}+\ln\tfrac{1}{\varepsilon}) e^{\tilde{t}} \geq e^{\ln\frac{1}{\varepsilon} - \ln\frac{1}{\varepsilon}}\}$$

$$\text{Let } \varepsilon \to 0^+ \overset{a.s.}{\Rightarrow} \{\sup_{\tilde{t}\in[0,T]} z e^{\tilde{t}} \geq 1\} = \{z \geq e^{-T}\} \quad , z \text{ defined in ⓑ}.$$

$$\Rightarrow P\{\lim_{\varepsilon\to 0^+} \tau_\varepsilon - \ln\tfrac{1}{\varepsilon} \leq T\} = P\{z \geq e^{-T}\} = \int_{e^{-T}}^{\infty} \frac{1}{\sqrt{2\pi\frac{1}{2}}} e^{-\frac{x^2}{2\cdot\frac{1}{2}}}\, dx$$

$$= \int_{e^{-T}}^{\infty} \frac{e^{-x^2}}{\sqrt{\pi}}\, dx$$

#2. (a). Given $\begin{cases} dX_t = -X_t dt + dW_t \\ X_0 = \xi \end{cases}$, WTS: $X_t = e^{-t}\xi + \int_0^t e^{s-t} dW_s$, $t \geq 0$.

Let $Z_t := e^t X_t$, then by applying Itô's Rule to $f(t,x) := e^t x \in C^{1,2}$, and assuming $dX_t = -X_t dt + dW_t$, $X_0 = \xi$, we have P-a.s. :

$$\begin{aligned} dZ_t &= e^t X_t dt + e^t dX_t \\ &= e^t X_t dt + e^t(-X_t dt + dW_t) \quad \text{by assumption} \\ &= e^t dW_t \end{aligned}$$

$$\Rightarrow \quad Z_t = Z_0 + \int_0^t e^s dW_s \quad , \quad t \in [0,+\infty)$$

$$e^t X_t = X_0 + \int_0^t e^s dW_s = \xi + \int_0^t e^s dW_s \quad , \quad t \in [0,+\infty)$$

$$X_t = e^{-t}\xi + \int_0^t e^{s-t} dW_s \quad , \quad t \in [0,+\infty)$$

An Alternative Method:

$\dot{\Phi}(t) = (-1)\Phi(t)$, $\Phi(0) = 1$ $\Rightarrow$ $\Phi(t) = e^{-t}$ is the fundamental sol to the deterministic homogeneous equation.

$\Rightarrow$ $X_t := \Phi(t)[X_0 + \int_0^t \Phi^{-1}(s) dW_s] = e^{-t}\xi + \int_0^t e^{s-t} dW_s$ solves the S.D.E.

It remains to prove $(X_t)_{t\geq 0}$ defined above is a strong solution and satisfies the uniqueness condition to conclude if $(\tilde{X}_t)_{t\geq 0}$ is an other sol, then $\tilde{X}_t = X_t = e^{-t}\xi + \int_0^t e^{s-t} dW_s$.

(i) $X_t = e^{-t}\left(\xi + \int_0^t e^s dw_s\right)$ is adapted to $\mathcal{F}_t$, where

$$\mathcal{F}_t := \sigma(\mathcal{G}_t \cup \mathcal{N}), \quad \begin{cases} \mathcal{G}_t := \sigma(\xi, w_s, 0 \le s \le t), \quad \mathcal{G}_\infty = \sigma\left(\bigcup_{t\ge 0} \mathcal{G}_t\right) \\ \mathcal{N} := \{ N \subset \Omega : \exists G \in \mathcal{G}_\infty \text{ s.t. } N \subset G, P(G) = 0 \} \end{cases}$$

(ii) $P\{X_0 = \xi\} = P\{\xi = \xi\} = 1$

(iii) Let $Y_t := \int_0^t e^s dw_s$, $\langle Y \rangle_t = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1)$.

Let $B_t \ge \langle Y \rangle_t$, then by exponential inequality we have, $\forall \varepsilon > 0$

$P\{\sup_{s\in[0,t]} |Y_s| \le a_\varepsilon\} \ge 1 - 2e^{-\frac{a_\varepsilon^2}{2B}} \ge 1-\varepsilon$ for a_ε large

$\Rightarrow P\{\sup_{s\in[0,t]} |Y_s| < +\infty\} = 1$ by letting $a_\varepsilon \to +\infty$.

$\Rightarrow P\{\int_0^t |e^{-s}(\xi + \int_0^s e^r dw_r)| ds < +\infty\} = 1$ as $\begin{cases} |\int_0^t e^{-s}\xi\, ds| \le \|\xi\|_\infty < \infty \\ |\int_0^t e^{-s} Y_s\, ds| \le \sup_{s\in[0,t]} |Y_s| < \infty \end{cases}$

(iv) X_t satisfies the S.D.E. by construction

By (i), (ii), (iii), (iv), X_t is a strong solution

Now, since $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| = |x-y| \le |x-y|$

we have the strong solution is unique.

$\Rightarrow X_t$ is the solution.

$\square$

#2. (b). WTS: Y^n converges in distribution to a limiting process Z, in the uniform topology on $C[0,T]$, and describe the distribution of Z, given $Y^n_t = X_{n+t}$, $t \in [0,T]$.

Proof: (i) Define $X^n_t := \int_0^{n+t} e^{s-(n+t)} dW_s$ on $t \in [0,T] \Rightarrow X^n = Y^n - e^{-(n+t)}\xi$, $\forall n \in \mathbb{N}$

Claim: $(X^n)_{n\in\mathbb{N}}$ is tight on $(C[0,T], \|\cdot\|_\infty)$

Proof: (by Kolmogorov - Chentsov)

$$E|X_n(0)|^2 = \int_0^n e^{2s-2n} ds = e^{-2n}\left(\tfrac{1}{2}(e^{2n}-1)\right) = \tfrac{1}{2}(1-e^{-2n}) \leq \tfrac{1}{2}$$

$$E|X_n(t)-X_n(s)|^2 = E\left(\int_0^{n+t} e^{r-(n+t)} dW_r - \int_0^{n+s} e^{r-(n+s)} dW_r\right)^3$$

$$= E\Big(\underbrace{\int_0^{n+s} e^{-n}e^{r}(e^{-t}-e^{-s}) dW_r}_{①} + \underbrace{\int_{n+s}^{n+t} e^{r-(n+t)} dW_r}_{②}\Big)^3$$

$$(*) = e^{-2(n+s)}(e^{-(t-s)}-1)^2\int_0^{n+s} e^{2r} dr + e^{-2(n+t)}\int_{n+s}^{n+t} e^{2r} dr + 3E\left(E(② \mid \mathcal{F}^W_{n+s})\, ①^2\right)$$
$$+3E\left(E(②^2 \mid \mathcal{F}^W_{n+s})\, ①\right)$$

$$= \left(|t-s|^2 e^{2T}\tfrac{1}{2}(1-e^{-2(n+s)})\right)^{\frac{3}{2}} + \left(\tfrac{1}{2}|t-s|e^{T}\right)^{\frac{3}{2}} + 0 + 0$$

$$\leq C_1|t-s|^3 + C_2|t-s|^2$$

$$\Rightarrow \begin{cases} E|X_n(0)|^\nu \leq \tfrac{1}{2}, & \nu = 2 \\ E|X_n(t)-X_n(s)|^\alpha \leq C_1|t-s|^{1+\beta_1} + C_2|t-s|^{1+\beta_2}, & \alpha=3,\ \beta_1=2,\ \beta_2=1 \end{cases}$$

$\Rightarrow (X^n)_{n\in\mathbb{N}}$ is tight.

□

(ii) Claim: $\|Y^n - X^n\|_\infty \xrightarrow{D} 0$

Proof: $\|Y^n - X^n\|_\infty = \sup_{t\in[0,T]} |e^{-(n+t)}\xi|$

$\leq e^{n}K$ P-a.s. since ξ is bounded P-a.s. by assumption

$\to 0$ as $n\to+\infty$

$\Rightarrow \|Y^n - X^n\|_\infty \xrightarrow[n\to+\infty]{P\text{-a.s.}} 0 \quad \Rightarrow \|Y^n - X^n\|_\infty \xrightarrow[n\to+\infty]{D} 0$

(iii) Claim: X^n converges to some limiting process Z on $C[0,T]$

Proof: Since X^n is tight by (i), it suffices to show convergence in finite dimensional distribution.

Let $\{t_1,\cdots,t_k\}\subset[0,T]$, w/ $t_0=0$, $t_{k+1}:=T$, we have:

$X^n_{t_j} = \int_0^{n+t_j} e^{s-(n+t_j)}\,dW_s \quad , \forall j\in\{0,\cdots,k+1\}$

$X^n_{t_j}$ is Gaussian by HW#3, Q#17, $\forall j\in\{0,\cdots,k+1\}$

Also, $X^n_{t_j} - X^n_{t_{j-1}} = \int_{n+t_{j-1}}^{n+t_j} e^{s-(n+t_j)}\,dW_s + \int_0^{n+t_{j-1}} e^{-n}(e^{-t_j} - e^{-t_{j-1}})e^{s}\,dW_s$.

$\Rightarrow$ The vector $(X^n_{t_j})_{j=0}^{k+1}$ can be written as $A(V^n_{t_j})_{j=1}^{k+1}$

for some matrix $A\in\mathbb{R}^{(k+1)\times(k+1)}$, and $V^n_{t_j} := \int_{n+t_{j-1}}^{n+t_j} e^{s-(n+t_j)}\,dW_s$

But $(V^n_{t_j})_{j=1}^{k+1}$ is an independent Gaussian vector.

$\Rightarrow$ By HW#1, Q#12, $(X^n_{t_j})_{j=1}^{k+1}$ is a Gaussian vector.

Now, $\forall j\in\{0,\cdots,k+1\}$

$E({X^n_{t_j}}^2) = \int_0^{n+t_j} e^{2s}\,ds\, e^{-2(n+t_j)} = \frac{1}{2}(1-e^{-2(n+t_j)}) \xrightarrow[n\to+\infty]{} \frac{1}{2}$

$\forall i\neq j\in\{0,\cdots,k+1\}$, and assume WOLG $i<j$

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$$E(X^n_{t_i}X^n_{t_j}) = \int_0^{n+t_i} e^{2s}\,ds\, e^{-2n} e^{-(t_i+t_j)} = \frac{1}{2}\left(e^{-(t_j-t_i)} - e^{-2n-(t_i+t_j)}\right) \xrightarrow[n\to+\infty]{} \frac{1}{2}e^{-(t_j-t_i)}$$

Since we assume $i<j$ WOLG, we have for general $i \neq j$,

$$E(X^n_{t_i}X^n_{t_j}) = \frac{1}{2}\left(e^{-|t_i-t_j|} - e^{-2n-(t_i+t_j)}\right)$$

Define $C^n \in \mathbb{R}$ by $C^n_{ij} := E(X^n_{t_i}X^n_{t_j})$, $\forall n \in \mathbb{N}$

$\Rightarrow V^n \sim \mathcal{N}(0, (C^n_{ij})_{\substack{0\le i\le k+1\\0\le j\le k+1}})$, where $C^n_{ij} \xrightarrow[n\to+\infty]{} C_{ij} = \frac{1}{2}e^{-|t_i-t_j|}$

$\Rightarrow \varphi_{V^n} = \exp(t^T(C^n_{ij})t) \xrightarrow[n\to+\infty]{\text{pointwise}} \exp(t^T(C_{ij})t)$, $(C_{ij})_{\substack{0\le i\le k+1\\0\le j\le k+1}} := \lim_{n\to+\infty}(C^n_{ij})_{\substack{0\le i\le k+1\\0\le j\le k+1}}$

Now, by Lévy's Thm, we have:

$\varphi_{V^n} \xrightarrow[n\to+\infty]{\text{pointwise}} \varphi_V \Rightarrow V^n \xrightarrow[n\to+\infty]{D} V$, $V \sim \mathcal{N}(0, (C_{ij})_{\substack{0\le i\le k+1\\0\le j\le k+1}})$

Since our choice of $(t_i)_{i=1}^k$ is arbitrary, we have the finite dimensional distribution of $(X^n)_{n\in\mathbb{N}}$ converges.

Since we showed $(X^n)_{n\in\mathbb{N}}$ is tight (i), we have $\exists$ Z on $(C[0,T], \|\cdot\|_\infty)$ s.t. $X^n \xrightarrow[n\to+\infty]{D} Z$. Since we also showed that $\|Y^n - X^n\|_\infty \xrightarrow[n\to+\infty]{D} 0$ in (ii), we have: $Y^n \xrightarrow[n\to+\infty]{D} Z$ (i.e. $P(Y^n)^{-1} \xrightarrow[n\to+\infty]{\text{weakly}} PZ^{-1}$).

$\square$

(iv) Since for arbitrary partition $\{t_i\}_{i=1}^k$, we have $(Z_{t_i})_{t=0}^{k+1} = \lim_{n\to+\infty} V^n \sim \mathcal{N}(0, C)$ we have Z is Gaussian w/
$$\begin{cases} E(Z_t) = 0, \ \forall t \\ E(Z_sZ_t) = \frac{1}{2}e^{-|t-s|}, \ \forall t,s \in [0,T] \end{cases}$$

$\Downarrow$ weakly stationary

$\Downarrow$ (Gaussian) stationary

$\Rightarrow Z_t = \frac{e^{-t}}{\sqrt{2}} W_{e^{2t}}$ is O-U process.

where W represents standard Wiener process.

#4. ⓐ. WTS : $u(t,x) = P\{\tau > t\}$, where $\tau := \inf\{s \geq 0, x + W_s \in \{0,1\}\}$.

Proof: Let $Y_s := u(t-s, x+W_s)$

Since $u \in C^{1,2}$, we can apply Itô:

$$dY_s = [-u_t(t-s, x+W_s) + \tfrac{1}{2} u_{xx}(t-s, x+W_s)]\,ds + u_x(t-s, x+W_s)\,dW_s$$

$$= u_x(t-s, x+W_s)\,dW_s$$ as u is sol to heat equation by assumption.

$$\Rightarrow Y_s = Y_0 + \int_0^s u_x(t-r, x+W_r)\,dW_r$$

Since $u_x(t-r, x+W_r)$ is continuous (differentiable) on $[0,T]$, for any T, we have $u_x(t-r, x+W_r) \in \mathcal{L}^*(W)$

$$\Rightarrow \left(\int_0^s u_x(t-r, x+W_r)\,dW_r\right)_{s\geq 0} =: \left(I(u_x)_s\right)_{s \geq 0} \in \mathcal{M}$$

$$\Rightarrow Y_0 = u(t,x) = E[Y_s - I(u_x)_s] = E\,Y_s - E\,I(u_x)_s$$

$$= E[u(t-s, x+W_s)]$$ as $\begin{cases} E\,I(u_x)_s = I(u_x)_0 = 0 \\ Y_s \overset{def}{=} u(t-s, x+W_s) \end{cases}$

Now, $u(t,x) = E^x u(t-s, W_s)$ and Helly-Bray Lemma implies (see (*))

$u(t,x) = \int_{-\infty}^{+\infty} p(t,x,y)\,dF(y)$ for some non-decreasing function $F: \mathbb{R} \to \mathbb{R}$.

where $p(t,x,y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$ is the fundamental solution.

(*). $u(t,x) = E^x u(s, W_{t-s}) = \int_{-\infty}^{+\infty} u(s,z)\, p(t-s, x, z)\,dz$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(s,z,y)\, p(t-s,x,z)\,dz\,dF(y)$$ for some $F: \mathbb{R} \to \mathbb{R}$ nondecreasing.

$$= \int_{-\infty}^{+\infty} p(t,x,y)\,dF(y)$$

From $u(t,x) = \int_{-\infty}^{+\infty} p(t,x,y)\, dF(y)$ and $\lim_{(s,y)\to(0,x)} u(s,y) = \underset{\uparrow \text{ initial condition}}{\mathbb{1}_{(0,1)}(x)}$, we can apply Widder's Uniqueness Thm to have $dF(y) = f(y)\,dy$, $f(y) := \mathbb{1}_{(0,1)}(y)$

$u(t,x) = \int_{-\infty}^{+\infty} p(t,x,y)\mathbb{1}_{(0,1)}(y)\,dy = E^x(\mathbb{1}_{(0,1)}(W_t))$ is the unique representation.

Note: representation is valid for $t \in (0,+\infty)$ since $\int_{-\infty}^{+\infty} e^{-ax^2}\mathbb{1}_{(0,1)}(x)\,dx < \infty$ even if $a=0$ $\Rightarrow$ $u(t,x) = \int_0^1 p(t,x,y)\mathbb{1}_{(0,1)}(y)\,dy$ is valid on $t \in (0, \frac{1}{2a} = +\infty)$, $x \in \mathbb{R}$.

By Boundary Condition (absorb): $\begin{cases} \lim_{(s,y)\to(t,1)} u(s,y) = u(t,1) = 0 \\ \lim_{(s,y)\to(t,0)} u(s,y) = u(t,0) = 0. \end{cases}$

$$\begin{aligned} u(t,x) &= E^x\left(\mathbb{1}_{(0,1)}(W_t)\mathbb{1}_{\{\tau > t\}} + 0\,\mathbb{1}_{\{t \geq \tau\}}\right) \\ &= E^x\left(\mathbb{1}_{(0,1)}(W_t)\mathbb{1}_{\{\tau > t\}}\right) \\ &= E^x\left(\mathbb{1}_{\{\tau > t\}}\right) \qquad \text{as } \{\tau > t\} \subset \{W_t \in (0,1)\} \\ &= P\{\tau > t\}. \end{aligned}$$

$\square$

#4 (b). Let $x = \frac{1}{2}$, WTS: $\exists\ a, b > 0$ s.t. $\frac{P\{\tau \geq t\}}{a e^{-bt}} \to 1$, as $t \to +\infty$

Proof: ①. solve the IBVP:

Let $u = f(t) g(x)$, then $f'(t) g(x) = \frac{1}{2} f(t) g''(x)$

$\Rightarrow 2 \frac{f'(t)}{f(t)} = \frac{g''(x)}{g(x)} =: \lambda$, $\quad$ (If $\lambda = 0$, $\begin{cases} f(t) \equiv f_0 \\ g(x) = c_1 + c_2 x \end{cases}$)

$\begin{cases} f(t) = f_0 e^{\frac{1}{2}\lambda t} \\ g(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x} \end{cases} \Rightarrow \begin{cases} \lambda \neq 0, & u = (c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x})(f_0 e^{\frac{1}{2}\lambda t}) \\ \lambda = 0, & u = (c_1 + c_2 x) f_0 \end{cases}$

$\sum_k a_k g_k(0) f_k(t) = 0$ $\quad$ - If $\lambda = 0$, $c_1 = c_2 = 0 \Rightarrow u \equiv 0$

$\Rightarrow$

$\sum_k a_k g_k(1) f_k(t) = 0$ $\quad$ If $\lambda \neq 0$, $c_1 = -c_2$, and

$c_1 (e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}) = 0 \Leftrightarrow \lambda = -k^2 \pi^2$

$k \in \mathbb{N}$

$\Rightarrow \begin{cases} \lambda = 0, & u \equiv 0 \quad \text{(violates initial condition)} \\ \lambda \neq 0, & u_k(t, x) = c_1 (e^{ik\pi x} - e^{-ik\pi x}) f_0 e^{-\frac{k^2 \pi^2}{2} t} \end{cases}$

$= B_k \sin(k\pi x) e^{-\frac{k^2\pi^2}{2} t}$, $\quad B_k := 2 i c_1 f_0$

Now, $u(t, x) = \sum_{k=1}^{+\infty} B_k \sin(k\pi x) e^{-\frac{k^2\pi^2}{2} t}$, $\quad u(0, x) = \mathbb{1}_{(0,1)}(x)$

$\Rightarrow \mathbb{1}_{(0,1)}(x) = \sum_{k=1}^{\infty} B_k \sin(k\pi x) \Rightarrow B_k = 2 \int_0^1 \sin(k\pi x)\, dx$

$= \frac{2}{k\pi}[1 - (-1)^k] = \begin{cases} \frac{4}{k\pi}, & k = 2i-1 \\ 0, & k = 2i \end{cases}$

$\Rightarrow u(t, x) = \sum_{i=1}^{+\infty} \frac{4}{(2i-1)\pi} \sin((2i-1)\pi x)\, e^{-\frac{(2i-1)^2 \pi^2}{2} t}$

②. find $a, b > 0$, given $x = \frac{1}{2}$.

$$u(t,x) = \sum_{i=1}^{+\infty} \frac{4}{(2i-1)\pi} \sin((2i-1)\pi x)\, e^{-\frac{(2i-1)^2\pi^2}{2} t}$$

$$u(t,\tfrac{1}{2}) = \sum_{i=1}^{+\infty} (-1)^{i-1} \frac{4}{(2i-1)\pi} e^{-\frac{(2i-1)^2\pi^2}{2} t}. \quad = P\{\tau > t\} \text{ by } ①.$$

$$\frac{P\{\tau > t\}}{a e^{-bt}} = \sum_{i=1}^{+\infty} \frac{\frac{(-1)^{i-1} 4}{(2i-1)\pi}}{a} e^{\left(b - \frac{(2i-1)^2\pi^2}{2}\right)t} \xrightarrow[t\to+\infty]{} 1$$

$$b = \frac{\pi^2}{2}, \quad a = \frac{4}{\pi} \Rightarrow \lim_{t\to+\infty} \frac{P\{\tau > t\}}{a e^{-bt}} = \frac{\frac{4}{\pi}}{a} + 0 = 1$$

for $b \in (0, \frac{\pi^2}{2})$, $\frac{P\{\tau \geq t\}}{a e^{-bt}} \xrightarrow[t\to+\infty]{} 0$

for $b \in (\frac{\pi^2}{2}, +\infty)$, $\frac{P\{\tau > t\}}{a e^{-bt}} \xrightarrow[t\to+\infty]{} \pm\infty$.

#5. WT: Use Donsker's F.C.L.T. to find the limiting distribution of $\frac{S_n^*}{\sqrt{n}}$, provided $S_n^* := \max\{0, S_1, \cdots, S_n\}$, $S_i := \sum_{k=1}^{i} X_k$, $\{X_k\}_{k\in\mathbb{N}}$ i.i.d. w/ $\begin{cases} EX_k = 0 \\ EX_k^2 = 1 \end{cases}$

Proof: ①. Define $f: D \to \mathbb{R}$ by $f(X) := \sup_{t\in[0,1]} \{X(t)\}$.

Claim: f is continuous from $(C[0,1], \|\cdot\|_\infty)$ to $\mathbb{R}$.

Proof: Pick a sequence $\{X_n\}_{n\in\mathbb{N}} \subset C[0,1]$ s.t. $\lim_{n\to\infty} \|X_n - X\|_\infty = 0$.

Then $f(X) = \sup_{t\in[0,1]} \{X(t)\}$ by definition

$$\leq \sup_{t\in[0,1]}\{X_n(t)\} + \sup_{t\in[0,1]} \{X_n(t) - X(t)\}$$

$$\leq f(X_n) + \|X_n - X\|_\infty$$

Similarly, $f(X_n) \leq f(X) + \|X_n - X\|_\infty$

$-\|X_n - X\|_\infty \leq |f(X) - f(X_n)| \leq \|X_n - X\|_\infty \Rightarrow \lim_{n\to\infty} |f(X) - f(X_n)| = 0$.

$\square$

②. Claim $X_n \xrightarrow[n\to\infty]{D} X$ implies $f(X_n) \xrightarrow[n\to\infty]{D} f(X)$, $X, X^n : [0,1] \times \Omega \to \mathbb{R}$

Proof: By Portmanteau Thm, $X_n \xrightarrow{D} X$ iff $\limsup_{n\to\infty} P(X_n \in F) \leq P(X \in F)$ for $\forall$ closed sets F.

Now, $P\{f(X_n) \in F\} = P\{X_n \in f^{-1}(F)\} \leq P\{X_n \in \overline{f^{-1}(F)}\}$

But $\limsup_{n\to\infty} P\{X_n \in \overline{f^{-1}(F)}\} \leq P\{X \in f^{-1}(F)\}$ by continuity of f

$\Rightarrow \limsup_{n\to\infty} P\{f(X_n) \in F\} \leq P\{f(X) \in F\}$, $\forall$ closed sets F

$\Rightarrow f(X_n) \xrightarrow{D} f(X)$ by Portmanteau Thm.

$\square$

- Finally, define $(Y^{(n)})_{n\in\mathbb{N}}$ by $Y^{(n)}_t := \frac{1}{\sqrt{n}}[S_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor) X_{\lfloor nt\rfloor+1}]$, $t\in[0,1]$

By the linear interpolation, we have $Y^{(n)} \in C[0,1]$, $\forall n\in\mathbb{N}$

Also, $\frac{S_n^*}{\sqrt{n}} = \sup_{t\in[0,1]} Y^{(n)}_t$, $\forall n\in\mathbb{N} \Rightarrow \frac{S_n^*}{\sqrt{n}} = f(Y^{(n)})$, $\forall n\in\mathbb{N}$.

where $f: C[0,1]\to\mathbb{R}$ is defined in ①.

Now, by Donsker's FCLT, we have $Y^{(n)} \xrightarrow{D} W$ on $C[0,1]$.

↑ standard 1-D Wiener Process

Apply ② to $(Y^{(n)})_{n\in\mathbb{N}}$, we have:

$$Y^{(n)} \xrightarrow{D} W \Rightarrow f(Y^{(n)}) \xrightarrow{D} f(W)$$

$$\Rightarrow \lim_{n\to\infty} \frac{S_n^*}{\sqrt{n}} \stackrel{D}{=} f(W) = \sup_{t\in[0,1]}\{W_t\}$$

□

$$P\left\{\lim_{n\to\infty}\frac{S_n^*}{\sqrt{n}} \geq c\right\} = P\left\{\sup_{t\in[0,1]}\{W_t\} \geq c\right\} = 2P\{W_1 \geq c\},\quad \forall c\in\mathbb{R}$$

$$= 2P\{N(0,1)\geq c\},\ \forall c\in\mathbb{R}.$$