Shizhou Xu HW#3. Limit Thms II.

#1. WTS: X_τ is a R.V. w.r.t. $\mathcal{F}_\tau$, provided $(X_t, \mathcal{F}_t)$ is cts. and τ is a stopping time w.r.t. $\mathcal{F}_t$.

Proof: Define $T : \{\tau \le t\} \to [0,t] \times \Omega$
$\omega \to (\tau(\omega), \omega)$

$T^{-1}(B) \in (\{\tau \le t\}, \mathcal{F}_t)$, $\forall B \in \mathcal{B}[0,t] \otimes \mathcal{F}_t \Rightarrow T$ is $(\{\tau\le t\}, \mathcal{F}_t)/\mathcal{B}[0,t]\otimes\mathcal{F}_t$ measurable.

Since $(X_t, \mathcal{F}_t)$ is continuous, we can approximate X_t by
$X_s^{(n)}(\omega) := X_{\frac{(k+1)t}{2^n}}(\omega)$, $s \in \left(\frac{kt}{2^n}, \frac{(k+1)t}{2^n}\right]$, which is progressively measurable

$\Rightarrow \lim_{n\to\infty} X_t^{(n)}(\omega) = X_t(\omega)$ by continuity, $\forall t, \forall \omega$

$\Rightarrow (X_t)$ is progressively measurable,

$X : (t,\omega) \to X_t(\omega)$ is $\mathcal{B}([0,t])\otimes\mathcal{F}_t / \mathcal{B}(\mathbb{R})$ measurable.

Now, $X_\tau : \omega \to X_\tau(\omega) = X \circ T$ is $(\{\tau\le t\}, \mathcal{F}_t) / \mathcal{B}(\mathbb{R})$ measurable.

$\Rightarrow \{X_\tau \in B\} \cap \{\tau \le t\} \in \mathcal{F}_t$, $\forall B \in \mathcal{B}(\mathbb{R})$

$\Rightarrow X_\tau$ is $\mathcal{F}_\tau / \mathcal{B}(\mathbb{R})$ measurable.

$\square$

#2. WTS: $(X_{t\wedge\tau}, \mathcal{F}_t)_{t\geq 0}$ is a mtg., given $(X_t, \mathcal{F}_t)$ is a cts. mtg. and τ is a stopping time w.r.t. $(\mathcal{F}_t)$.

Proof: ①. $X_{t\wedge\tau} = \mathbb{1}_{\{\tau\leq t\}} X_\tau + \mathbb{1}_{\{\tau > t\}} X_t$

Since $\begin{cases} X_\tau^{-1}(B) \cap \{\tau \leq t\} \in \mathcal{F}_t & \text{by \#1. HW\#3} \\ X_t^{-1}(B) \cap \{\tau > t\} \in \mathcal{F}_t & \text{by adaptedness and stopping time def.} \end{cases}$

we have $X_{t\wedge\tau}$ is $\mathcal{F}_t$-measurable.

②. Let $s \leq t$, then we have:

$$\begin{aligned} E(X_{t\wedge\tau} \mid \mathcal{F}_s) &= \underbrace{E(\mathbb{1}_{\{\tau\leq s\}} X_\tau \mid \mathcal{F}_s)}_{= \mathbb{1} X_\tau} + \underbrace{E(\mathbb{1}_{\{\tau\leq s\}}(X_t - X_s) \mid \mathcal{F}_s)}_{=0} \\ &\quad + \underbrace{E(\mathbb{1}_{\{\tau > s\}} X_s \mid \mathcal{F}_s)}_{= \mathbb{1}_{\{\tau > s\}} X_s} + \underbrace{E(\mathbb{1}_{\{s<\tau\leq t\}}(X_\tau - X_s) \mid \mathcal{F}_s)}_{=0} \\ &\quad + \underbrace{E(\mathbb{1}_{\{\tau > t\}}(X_t - X_s) \mid \mathcal{F}_s)}_{=0} \\ &= \mathbb{1}_{\{\tau\leq s\}} X_\tau + \mathbb{1}_{\{\tau > s\}} X_s = X_{\tau\wedge s} \end{aligned}$$

③. Since $(X_t, \mathcal{F}_t)$ is a mtg. $E|X_t| < \infty$, $\forall t$,

Now, $E|X_{t\wedge\tau}| \leq \sup_{0\leq s\leq t} E|X_t| < \infty$.

By ①, ②, ③, $(X_{t\wedge\tau})$ is a mtg. w.r.t. $(\mathcal{F}_t)$. $\square$

#3. Given $(P_x)_{x\in\mathbb{R}^d}$ w/ $P(\cdot,\cdot,\cdot)$ satisfying

$$P(t,x,B_\varepsilon^c(x)) = o(t) \qquad (*)$$

$$\int_{B_\varepsilon(x)} (y^i - x^i)\, P(t,x,dy) = b^i(x)\, t + o(t) \quad (**) \qquad t\to 0 \text{ uniformly}$$

$$\int_{B_\varepsilon(x)} (y^i - x^i)(y^j - x^j)\, P(t,x,dy) = a^{ij}(x)\, t + o(t) \quad (***)$$

WTS: If $f:\mathbb{R}^d\to\mathbb{R}$ that is bdd & have bdd & uniformly cts 1^{st}, 2^{nd} partial derivatives, Af is defined and $Af = \sum_{i=1}^d b^i \partial_i f + \frac{1}{2}\sum_{i,j=1}^d a^{ij}\partial_{ij} f$

Proof: Pick $x\in\mathbb{R}^d$ arbitrarily. By Taylor expansion: $\tilde{y}\in\{tx+(1-t)y \mid t\in[0,1]\}$

$$f(y) = f(x) + \underbrace{\sum_{i=1}^d \partial_i f(x)(y^i - x^i)}_{①} + \underbrace{\frac{1}{2}\sum_{i,j=1}^d \partial_{ij} f(\tilde{y})(y^i - x^i)(y^j - x^j)}_{②}$$

Also, pick $\varepsilon>0$. we have

$$\int_{\mathbb{R}^d} f(y) P(t,x,dy) = \int_{B_\varepsilon^c} f(y) P(t,x,dy) + \int_{B_\varepsilon} f(y) P(t,x,dy)$$

(i) $\Rightarrow$ $\int_{B_\varepsilon^c(x)} f(y) P(t,x,dy) - \int_{B_\varepsilon^c(x)} f(x) P(t,x,dy) \le K\, o(t) \xrightarrow[t\to 0]{} 0$ by $(*)$ and bddness of f

$$\int_{B_\varepsilon(x)} f(y) P(t,x,dy) - \int_{B_\varepsilon(x)} f(x) P(t,x,dy) = \int_{B_\varepsilon(x)} (① + ②) P(t,x,dy)$$

①: If $i\in\{1,\dots,d\}$.

$$\int_{B_\varepsilon(x)} \partial_i f(x)(y^i - x^i) P(t,x,dy) = \partial_i f(x)\big(t b^i(x) + o(t)\big) \quad \text{by } (**)$$
$$= t\,\partial_i f(x)\, b^i(x) + o(t) \quad \text{by bddness of } \partial_i f$$

(ii) $\Rightarrow$ $\int_{B_\varepsilon(x)} ①\, P(t,x,dy) = t\sum_{i=1}^d b^i(x)\,\partial_i f(x) + o(t)$

② : $\forall\, i,j \in \{1,\dots,d\}$,

$$\int_{B_\varepsilon(x)} \partial_{ij} f(\tilde{y})(y^i - x^i)(y^j - x^j)\, P(t,x,dy) = \partial_{ij} f(\tilde{y})\left(t a^{ij}(x) + o(t)\right) \text{ by (****)}.$$

$$= t\, \partial_{ij} f(\tilde{y})\, a^{ij}(x) + o(t) \text{ by bddness of } \partial_{ij} f$$

(iii) $\Rightarrow$ $\displaystyle\int_{B_\varepsilon(x)} ② \, P(t,x,dy) = \frac{t}{2} \sum_{i,j=1}^{d} a^{ij}(x)\, \partial_{ij} f(\tilde{y}) + o(t)$

Since this is true for $\forall\, \varepsilon > 0$, let $\varepsilon \to 0$, we have:

(iv) $\tilde{y} \in B_{|y-x|}(x) \subset B_\varepsilon(x) \Rightarrow \tilde{y} \to x \Rightarrow \partial_{ij} f(\tilde{y}) \to \partial_{ij} f(x)$, $\forall\, i,j$ by the uniform continuity of $\partial_{ij} f$.

Finally, we have

$$\int_{\mathbb{R}^d} f(y) P(t,x,dy) - f(x) = \int_{B_\varepsilon(x)} ① \, P(t,x,dy) + \int_{B_\varepsilon(x)} ② \, P(t,x,dy) + \overset{\text{by (i)}}{K}\, o(t)$$

$$= t \sum_{i=1}^{d} b^i(x)\, \partial_i f(x) + \frac{t}{2} \sum_{i,j=1}^{d} a^{ij}(x)\, \partial_{ij} f(\tilde{y}) + K\, o(t) \text{ by (ii)(iii)}$$

$$\underset{\varepsilon \to 0}{\text{let}} \quad = t \sum_{i=1}^{d} b^i(x)\, \partial_i f(x) + \frac{t}{2} \sum_{i,j=1}^{d} a^{ij}(x)\, \partial_{ij} f(x) + K\, o(t) \text{ by (iv)}$$

$$Af(x) = \lim_{t \to 0} \frac{\int_{\mathbb{R}^d} f(y) P(t,x,dy) - f(x)}{t} = \lim_{t \to 0} \sum_{i=1}^{d} b^i(x)\, \partial_i f(x) + \frac{1}{2} \sum_{i,j}^{d} a^{ij}(x)\, \partial_{ij} f(x) + \frac{K\, o(t)}{t}$$

$$= \sum_{i=1}^{d} b^i(x)\, \partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x)\, \partial_{ij} f(x)$$

Since our choice of $x \in \mathbb{R}^d$ is arbitrary, and the proof is independent of x due to uniform continuity, bddness, and convergence, we have:

$$Af = \sum_{i=1}^{d} b^i \partial_i f + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij} \partial_{ij} f$$

$\square$

#4. WTS: The infinitesimal generator of (X_t) by $\begin{cases} X_1(t) = X_1(0) + W_t \\ X_2(t) = X_2(0) + \int_0^t X_1(s)\,ds \end{cases}$

on $D_A := \{ f : \mathbb{R}^2 \to \mathbb{R} \in C^2_\infty(\mathbb{R}^2). \}$.

Let $f \in D_A$, we have

$$b_1^t := \frac{1}{t} E\left(\mathbb{1}_{B_\varepsilon(x)}[X_1(t) - X_1(0)]\right) = \frac{1}{t} E\left(\mathbb{1}_{B_\varepsilon(x)} W_t\right) = 0$$

$$b_2^t := \frac{1}{t} E\left(\mathbb{1}_{B_\varepsilon(x)}[X_2(t) - X_2(0)]\right) = \frac{1}{t} E\left(\mathbb{1}_{B_\varepsilon(x)}\left[X_1(0)\,t + \int_0^t W_s\,ds\right]\right) = x_1 + o(t)$$

$$a_{21}^t = a_{12}^t := \frac{1}{t} E\left(\mathbb{1}_{B_\varepsilon(x)}\left[W_t X_1(0)\,t + W_t \int_0^t W_s\,ds\right]\right) = o(t)$$

$$a_{11}^t := \frac{1}{t} E\left(\mathbb{1}_{B_\varepsilon(x)}[W_t^2]\right) = 1$$

$$a_{22}^t := \frac{1}{t} E\left(\mathbb{1}_{B_\varepsilon(x)}\left[X_1(0)^2 t^2 + 2X_1(0)\,t\int_0^t W_s\,ds + \left(\int_0^t W_s\,ds\right)^2\right]\right) = o(t)$$

Let $a_{ij} := \lim_{t \to 0^+} a_{ij}^t$, $b_i := \lim_{t\to 0^+} b_i^t$, we have that

$$Af(x) = x_1 \frac{\partial}{\partial x_2} f(x) + \frac{1}{2} \frac{\partial^2}{\partial x_1^2} f(x), \quad w/ \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

#5. Find Af, if $f \in \{\mathbb{Z} \to \mathbb{R} : \text{bdd}\}$, given $P(i,t,\{j\}) = e^{-\lambda t}\dfrac{(\lambda t)^{j-i}}{(j-i)!}$

Let $f : \mathbb{Z} \to \mathbb{R}$ be bdded, then

$$\frac{1}{t}\left(Ef(i) - f(i)\right) = \left(\sum_{j=i}^{\infty} P(i,t,\{j\}) f(j) - f(i)\right)\frac{1}{t}$$

$$= \sum_{j=i+1}^{\infty} e^{-\lambda t}\frac{(\lambda)^{j-i}(t)^{j-i-1}}{(j-i)!} f(j) + (e^{-\lambda t} - 1)\frac{1}{t} f(i)$$

$$= e^{-\lambda t}(\lambda) f(i+1) + \sum_{j=i+2}^{\infty} e^{-\lambda t}\frac{(\lambda)^{j-i}(t)^{j-i-1}}{(j-i)!} f(j) + f(i)\frac{(e^{-\lambda t} - 1)}{t}$$

$$Af(i) = \lim_{t \to 0^+} \frac{1}{t}\left(Ef(i) - f(i)\right)$$

$$= \underbrace{\lim_{t \to 0^+} e^{-\lambda t}\lambda f(i+1)}_{①} + \underbrace{\lim_{t \to 0^+} \sum_{j=i+2}^{\infty} e^{-\lambda t}\frac{\lambda^{j-i}(t)^{j-i-1}}{(j-i)!} f(j)}_{②} + \underbrace{\lim_{t \to 0^+} f(i)\frac{(e^{-\lambda t} - 1)}{t}}_{③}$$

$① = \lambda f(i+1)$

$② = \displaystyle\sum_{j=i+2}^{\infty} \lim_{t \to 0^+} e^{-\lambda t}\frac{\lambda^{j-i}(t)^{j-i-1}}{(j-i)!} f(j)$ by bdd convergence thm

$= 0$ $\qquad \displaystyle\sum_{j=i+2}^{\infty} e^{-\lambda t}\frac{\lambda^{j-i}(t)^{j-i-1}}{(j-i)!} f(j) \le \|f\|_\infty \cdot 1$

$③ = f(i)\displaystyle\lim_{t \to 0^+}\frac{(-\lambda)e^{-\lambda t}}{1} = f(i)(-\lambda)$ by L'hôpital's Rule

By ①, ②, ③

$$Af(i) = \lambda\left(f(i+1) - f(i)\right)$$

#6. WT: ①. Find the T.P.F. of the Ornstein-Uhlenbeck process.
②. Find the infinitesimal generator of the O-U process.

①. Since $X_t = e^{-t/2} W_{e^t}$ and $P(W_t \le y \mid W_s = x) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(u-x)^2}{2(t-s)}} du$

we have:

$$P(X_t \le y \mid X_s = x) = P(W_{e^t} \le y e^{t/2} \mid W_{e^s} = x e^{s/2})$$

$$= \int_{-\infty}^{y e^{t/2}} \frac{1}{\sqrt{2\pi(e^t - e^s)}} e^{-\frac{(u - x e^{s/2})^2}{2(e^t - e^s)}} du$$

Let $\hat{u} = e^{-t/2} u$, $du = e^{t/2} d\hat{u}$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(1-e^{-(t-s)})}} e^{-\frac{(\hat{u} - e^{-(t-s)/2} x)^2}{2(1-e^{-(t-s)})}} d\hat{u}$$

$$\Rightarrow P_X(x, t, \Gamma) = P(X_{s+t} \in \Gamma \mid X_s = x) = \int_{\Gamma} \frac{1}{\sqrt{2\pi(1-e^{-t})}} e^{-\frac{(y - e^{-t/2} x)^2}{2(1-e^{-t})}} dy$$

②. $X_t = e^{-\frac{t}{2}} X_0 + \int_0^t e^{-\frac{1}{2}(t-s)} dW_s$, where $X_0 \sim N(0,1)$, $X_0 \perp W_t$

$$E(X_t, X_s) = \underset{①}{E}\left(e^{-\frac{t}{2}} e^{-\frac{s}{2}} X_0^2\right) + \underset{②}{E}\left(e^{-\frac{s}{2}} X_0 \int_0^t e^{-\frac{1}{2}(t-r)} dW_r\right) + \underset{③}{E}\left(e^{-\frac{t}{2}} X_0 \int_0^s e^{-\frac{1}{2}(s-r)} dW_r\right)$$
$$+ \underset{④}{E}\left(\int_0^t e^{-\frac{1}{2}(t-r)} dW_r \int_0^s e^{-\frac{1}{2}(s-r)} dW_r\right)$$

①. $= e^{-\frac{t+s}{2}}$ as $E(X_0^2) = 1$

②. $=$ ③. $= 0$ as $X_0 \perp \sigma(W_t)$

④. $= E\left(\int_0^s e^{-\frac{1}{2}(t-r)} dW_r \int_0^s e^{-\frac{1}{2}(s-r)} dW_r\right) + E\left(\int_0^s e^{-\frac{1}{2}(s-r)} dW_r \int_s^t e^{-\frac{1}{2}(t-r)} dW_r\right)$

$= e^{-\frac{1}{2}(t-s)} E\left(\int_0^s e^{-\frac{1}{2}(s-r)} dW_r\right)^2 + e^{-\frac{1}{2}(t-s)} E\left(\int_0^s e^{-\frac{1}{2}(s-r)} dW_r \int_s^t e^{-\frac{1}{2}(s-r)} dW_r\right)$

Since $e^{-\frac{1}{2}(s-r)} \in \mathcal{L}^*$ trivially, $W \in \mathcal{M}_2^c$, $\int_0^t e^{-\frac{1}{2}(s-r)} dW_r \in \mathcal{M}_2^c$ w.r.t. $\mathcal{F}_t^W$.

$$\Rightarrow \begin{cases} E\left(\int_0^s e^{-\frac{1}{2}(s-r)} dW_r\right)^2 = E \int_0^s e^{-(s-r)} d\langle W \rangle_r = e^{-s} \int_0^s e^r dr = 1 - e^{-s} \\ E\left[\int_0^s e^{-\frac{1}{2}(s-r)} dW_r \, E\left(\int_s^t e^{-\frac{1}{2}(s-r)} dW_r \mid \mathcal{F}_s\right)\right] = 0 \end{cases}$$

$\Rightarrow$ ④. $= e^{-\frac{1}{2}(t-s)}(1 - e^{-s}) = e^{-\frac{1}{2}(t-s)} - e^{-\frac{1}{2}(t+s)}$

$$\Rightarrow E(X_t X_s) = e^{-\frac{1}{2}(t+s)} + e^{-\frac{1}{2}(t-s)} - e^{-\frac{1}{2}(t+s)} = e^{-\frac{1}{2}(t-s)}.$$

$X_t = e^{-\frac{t}{2}} X_0 + \int_0^t e^{-\frac{1}{2}(t-r)} dW_r$ is indeed O-U process

$$dX_t = -\frac{1}{2} e^{-t/2} X_0 dt + dW_t$$

By Itô's formula, for $f \in C^2$, we have:

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s) d\langle X\rangle_s$$

$$= -\frac{1}{2}\int_0^t f'(X_s) e^{-s/2} X_0 ds + \int_0^t f'(X_s) dW_s + \frac{1}{2}\int_0^t f''(X_s) ds$$

$$E_x f(X_t) - f(x) = -\frac{1}{2}\int_0^t E f'(X_s) e^{-s/2} x ds + E\int_0^t f'(X_s) dW_s + \frac{1}{2}\int_0^t E f''(X_s) ds$$

$$= -\frac{1}{2}\int_0^t E f'(X_s) e^{-s/2} x ds + \frac{1}{2}\int_0^t E f''(X_s) ds$$

$$\lim_{t\to 0^+} \frac{E_x f(X_t) - f(x)}{t} = \left(-\frac{1}{2}\right) f'(x) x + \frac{1}{2} f''(x) \quad \text{as} \quad \lim_{t\to 0^+} E f'(X_s) = E f'(X_0) = f'(x)$$

$$= -\frac{x}{2} f'(x) + \frac{1}{2} f''(x)$$

$$\lim_{t\to 0^+} e^{-t/2} = e^{-0} = 1$$

$$\lim_{t\to 0^+} E f''(X_s) = E f''(X_0) = f''(x)$$

#7. Given W^1, W^2 independent Wiener w/ filtration $(\mathcal{F}_t)_{t\geq 0}$, $0=t_0\leq t_1\leq\cdots\leq t_n=T$, $Q(t):=\sum_j X_{t_j}(W^1_{t_{j+1}}-W^1_{t_j})(W^2_{t_{j+1}}-W^2_{t_j})$

where X is a bdd process adapted to $(\mathcal{F}_t)_{t\geq 0}$.

WTS: $\lim_{\max(t_{j+1}-t_j)\to 0} Q(t) \overset{L^2}{=} 0$

Proof: Let $0=t_0\leq t_1\leq\cdots\leq t_n=T$ be an arbitrary partition of $[0,T]$.

$$E\left(Q(t)^2\right) = E\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} X_{t_i}X_{t_j}(W^1_{t_{i+1}}-W^1_{t_i})(W^2_{t_{i+1}}-W^2_{t_i})(W^1_{t_{j+1}}-W^1_{t_j})(W^2_{t_{j+1}}-W^2_{t_j})$$

For any $i<j$, we have

$$E\, X_{t_i}X_{t_j}(W^1_{t_{i+1}}-W^1_{t_i})(W^2_{t_{i+1}}-W^2_{t_i})(W^1_{t_{j+1}}-W^1_{t_j})(W^2_{t_{j+1}}-W^2_{t_j})$$

$$= E\left\{X_{t_i}\, E\left[X_{t_j}(W^1_{t_{i+1}}-W^1_{t_i})(W^2_{t_{i+1}}-W^2_{t_i})\, E\left((W^1_{t_{j+1}}-W^1_{t_j})(W^2_{t_{j+1}}-W^2_{t_j})\,|\,\mathcal{F}_{t_j}\right)\,|\,\mathcal{F}_{t_i}\right]\right\}$$

by tower property and the fact that $\begin{cases} X_{t_j},\ (W^{1,2}_{t_{i+1}}-W^{1,2}_{t_i}) \text{ are } \mathcal{F}_{t_j}\text{-measurable} \\ X_{t_i} \text{ is } \mathcal{F}_{t_i}\text{-measurable}.\ \mathcal{F}_{t_i}\subset\mathcal{F}_{t_j}\end{cases}$

But $E\left((W^1_{t_{j+1}}-W^1_{t_j})(W^2_{t_{j+1}}-W^2_{t_j})\,|\,\mathcal{F}_{t_j}\right) = E\left(W^1_{t_{j+1}}-W^1_{t_j}\,|\,\mathcal{F}_{t_j}\right)E\left(W^2_{t_{j+1}}-W^2_{t_j}\,|\,\mathcal{F}_{t_j}\right)$

$= 0$ by independence and W is Wiener

By similar argument, the conclusion is true for $j<i$.

$$\Rightarrow E\left(Q(t)^2\right) = E\left(\sum_{i=0}^{n-1} X^2_{t_i}(W^1_{t_{i+1}}-W^1_{t_i})^2(W^2_{t_{i+1}}-W^2_{t_i})^2\right)$$

$$= \sum_{i=0}^{n-1} E\left[X^2_{t_i}\, E\left((W^1_{t_{i+1}}-W^1_{t_i})^2(W^2_{t_{i+1}}-W^2_{t_i})^2\,|\,\mathcal{F}_{t_i}\right)\right],\ \text{as } X^2_{t_i} \text{ is } \mathcal{F}_{t_i}\text{-measurable}$$

$$= \sum_{i=0}^{n-1} E\left[X^2_{t_i}\, E\left((W^1_{t_{i+1}}-W^1_{t_i})^2\,|\,\mathcal{F}_{t_i}\right)E\left((W^2_{t_{i+1}}-W^2_{t_i})^2\,|\,\mathcal{F}_{t_i}\right)\right],\ \text{by independence}$$

$$= \sum_{i=0}^{n-1} E\left[X^2_{t_i}(t_{i+1}-t_i)^2\right]$$

Since X is a bdd process, $\exists K \in \mathbb{R}$ s.t. $E(X_t^2) \le K^2$, $\forall t \ge 0$.

$$E\left(Q(t)^2\right) = \sum_{i=0}^{n-1} E\left(X_{t_i}^2 (t_{i+1} - t_i)^2 \right)$$

$$= \sum_{i=0}^{n-1} E(X_{t_i}^2)(t_{i+1} - t_i)^2$$

$$\le \left(\sum_{i=0}^{n-1} E(X_{t_i}^2)(t_{i+1} - t_i) \right) \sup_i (t_{i+1} - t_i) \quad , \text{let } \bar{\Delta} t_i := \sup_i (t_{i+1} - t_i)$$

$$\Rightarrow \lim_{\bar{\Delta} t_i \to 0} E\left(Q(t)^2\right) \le \lim_{\bar{\Delta} t_i \to 0} \left(\sum_{i=0}^{n-1} E(X_{t_i}^2)(t_{i+1} - t_i) \right) \lim_{\bar{\Delta} t_i \to 0} \bar{\Delta} t_i$$

$$\le \left(\int_0^T K^2 \, dt \right) \lim_{\bar{\Delta} t_i \to 0} \bar{\Delta} t_i = 0$$

$$\Rightarrow \lim_{\max_i (t_{i+1} - t_i) \to 0} Q(t) \overset{L^2}{=} 0$$

$\square$

#8. Given $A_t + B_t W_t = 0$ $\forall t$, $(A_t, \mathcal{F}_t)$ & $(B_t, \mathcal{F}_t)$ have C^1 trajectory

WTS: $A \equiv B \equiv 0$

Proof: Let $t = 0$, we have $A_0 = 0$

Now, pick $t \in [0, +\infty)$ be arbitrary, $\forall \omega \in \Omega_1$, we have

$A_t(\omega) + B_t(\omega) W_t(\omega) = 0$

$$A_{t+h}(\omega) - A_t(\omega) = B_{t+h}(\omega) W_{t+h}(\omega) - B_t(\omega) W_t(\omega), \quad \forall h \geq 0.$$

$$= \left(B_{t+h}(\omega) - B_t(\omega)\right) W_t(\omega) + B_{t+h}(\omega)\left(W_{t+h}(\omega) - W_t(\omega)\right)$$

Time $\frac{1}{h}$ on both sides, we have

$$\lim_{h \to 0^+} \frac{A_{t+h}(\omega) - A_t(\omega)}{h} = \lim_{h \to 0^+} \frac{B_{t+h}(\omega) - B_t(\omega)}{h} W_t(\omega) + \lim_{h \to 0^+} B_{t+h}(\omega) \frac{W_{t+h}(\omega) - W_t(\omega)}{h}$$

$$\Rightarrow A_t'(\omega) = B_t'(\omega) W_t(\omega) + B_t(\omega) \lim_{h \to 0^+} \frac{W_{t+h}(\omega) - W_t(\omega)}{h}$$

Since $t \to A_t'(\omega)$, $t \to B_t'(\omega)$, $t \to W_t(\omega)$ $\in C^0([0, +\infty))$ by assumption.

If $B_t(\omega) \neq 0$, then $\lim_{h \to 0^+} \frac{W_{t+h}(\omega) - W_t(\omega)}{h} \in C^0([0, +\infty))$.

But (W_t) is Wiener $\Rightarrow$ nowhere differentiable $\rightarrow\leftarrow$ (contradiction)

$\Rightarrow B_t(\omega) = 0$

Since our choice of $t \in [0, +\infty)$, $\omega \in \Omega_1$ is arbitrary, $B \equiv 0$

$\Rightarrow B_t'(\omega) = 0$, $\forall t$, $\forall \omega$ $\Rightarrow$ $A_t'(\omega) = 0$ $\forall t$, $\forall \omega$

Since $A_0 = 0$, $A \equiv C$, we have $A \equiv 0$.

$\square$

#9. WTS: Given $f \in L^2([0,T], \mathcal{B}, \text{Leb})$, $f_h := \begin{cases} 0 & , t \in [0,h) \\ \frac{1}{h}\int_{(k-1)h}^{kh} f(s)\,ds & , t \in [kh, (k+1)h \wedge T) \end{cases}$, $\forall h > 0$

① $f_h \xrightarrow[h\to 0]{L^2} f$ ② $\|f_h\|_{L^2} \le \|f\|$

Proof: ① Claim: It suffices to show that $f_h \xrightarrow{L^2} f$, $\forall f \in C([0,T])$.

Indeed, if $f_h \xrightarrow{L^2} f$, $\forall f \in C([0,T])$, let $\{f_n\}_{n\in\mathbb{N}} \subset C([0,T])$ s.t. $f_n \xrightarrow{L^2} f$

First, for $\varepsilon > 0$, $\exists N$ s.t. if $n \ge N$, $\|f_n - f\|_{L^2} < \varepsilon/3$. Pick $\hat{n} \ge N$.

Now, pick h small enough s.t. $\|f_{\hat{n}h} - f_{\hat{n}}\|_{L^2} < \varepsilon/3$, which was possible by claim

$$\|f_{nh} - f_h\|_{L^2}^2 = \int_0^T \Big| \sum_{k=1}^{n(h)-1} \mathbb{1}_{[kh,(k+1)h\wedge T)} \tfrac{1}{h}\int_{(k-1)h}^{kh} (f_n - f)\,ds \Big|^2 dt$$

$$= \int_0^T \sum_{k=1}^{n(h)-1} \mathbb{1}_{[kh,(k+1)h\wedge T)} \tfrac{1}{h^2} \Big| \int_{(k-1)h}^{kh} (f_n - f)\,ds \Big|^2 dt$$

$$\le \sum_{k=1}^{n(h)-1} \Big[\Big(h \cdot \tfrac{1}{h^2} \int_{(k-1)h}^{kh} (f_n - f)\,ds \Big) \int_{(k-1)h}^{kh} (f_n - f)\,ds \Big]$$

$$\le \sum_{k=1}^{n(h)-1} \int_{(k-1)h}^{kh} |f_n - f|^2\,ds$$

$$\le \|f_n - f\|_{L^2}^2 < \left(\tfrac{\varepsilon}{3}\right)^2 < \tfrac{\varepsilon}{3}$$

$$\|f_h - f\|_{L^2} \le \|f - f_n\|_{L^2} + \|f_n - f_{nh}\|_{L^2} + \|f_{\hat{n}h} - f_h\|_{L^2} < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we have $\|f_h - f\|_{L^2}^2 \xrightarrow[h\to 0]{} 0 \Rightarrow f_h \xrightarrow[h\to 0]{L^2} f$

It remains to show that $f_h \xrightarrow{L^2} f$, $\forall f \in C[0,T]$.

But $$\|f_h - f\|_{L^2} = \int_0^T \Big| f\mathbb{1}_{[0,h)} + \sum_{k=1}^{n(h)-1} \mathbb{1}_{[kh,(k+1)h\wedge T)} \tfrac{1}{h}\int_{(k-1)h}^{kh} (f_n - f)\,ds \Big|^2 dt$$

$$\le \int_{[0,h)} |f|^2\,ds + \sum_{k=1}^{n(h)-1} \Big[\int_{kh}^{(k+1)h} |f|^2\,ds - 2\int_{(k-1)h}^{kh} f(s)\,ds \int_{kh}^{(k+1)h} f(s)\,ds + \int_{(k-1)h}^{kh} |f|^2\,ds \Big]$$

$$\le \int_{[0,h)} |f|^2\,ds + \Big[\sum_{k=1}^{n(h)-1} \Big(\int_{kh}^{(k+1)h} f(s)\,ds + \int_{(k-1)h}^{kh} f(s)\,ds \Big) \Big] \max_{\substack{1\le k\le n(h)-1 \\ x \in [(k-1)h,(k+1)h]}} \{|f(x) - f(y)|\}$$

Since $\lim_{h\to 0} \int_{[0,h)} |f|^2\,ds = 0$ by bdd convergence, and $\lim_{h\to 0} \Delta f(h) = 0$, we have:

$\lim_{h\to 0} \|f_h - f\|_{L^2} = 0$, $\forall f \in C([0,T])$ $\square$

②、 $\|\xi_h\|_{L^2}^2 = \int_0^T \left|\sum_{k=1}^{n(h)-1} \mathbb{1}_{[kh,(k+1)h\wedge T)}\frac{1}{h}\int_{(k-1)h}^{kh}\xi(s)ds\right|^2 dt$

$$= \int_0^T \sum_{k=1}^{n(h)-1} \mathbb{1}_{[kh,(k+1)h\wedge T)}\frac{1}{h^2}\left|\int_{(k-1)h}^{kh}\xi(s)ds\right|^2 dt$$

$$= \frac{1}{h}\sum_{k=1}^{n(h)-1}\left|\int_{(k-1)h}^{kh}\xi(s)ds\right|^2$$

$$= \sum_{k=1}^{n(h)-1}\int_{(k-1)h}^{kh}\xi(s)ds\left(\frac{1}{h}\int_{(k-1)h}^{kh}\xi(s)ds\right)$$

$$\leq \sum_{k=1}^{n(h)-1}\int_{(k-1)h}^{kh}|\xi(s)|^2 ds$$

$$\leq \int_0^h|\xi|^2 ds + \sum_{k=1}^{n(h)-1}\int_{kh}^{(k+1)h\wedge T}|\xi|^2 ds = \|\xi\|_{L^2}$$

$\square$

#10. WTS: $\int_0^T w^2(t)\,dw(t)$ from $\sum_{k=0}^{n(T)-1} w^2(t_k)(w(t_{k+1})-w(t_k))$

Denote $w_{t_{k+1}}-w_{t_k} =: \Delta w_{t_k}$, $t_{k+1}-t_k =: \Delta t_k$, $\forall k\in\{0,\dots,n(T)-1\}$

$$\sum_{k=0}^{n(T)-1} w_{t_k}^2(w_{t_{k+1}}-w_{t_k}) = \sum_{k=0}^{n(T)-1}\left[(w_{t_{k+1}}^3-w_{t_k}^3)-(w_{t_{k+1}}-w_{t_k})^3-3w_{t_k}(w_{t_{k+1}}-w_{t_k})^2\right]\frac{1}{3}$$

$$=\Bigg(\underbrace{\sum_{k=0}^{n(T)-1}(w_{t_{k+1}}^3-w_{t_k}^3)}_{①}-\underbrace{\sum_{k=0}^{n(T)-1}(\Delta w_{t_k})^3}_{②}-3\underbrace{\sum_{k=0}^{n(T)-1} w_{t_k}(\Delta w_{t_k})^2}_{③}\Bigg)\frac{1}{3}.$$

$$① = w_{t_{n(T)}}^3 - w_{t_0}^3 = w_T^3$$

$$E(②^2) = E\left(\sum_{i=0}^{n(T)-1}\sum_{j=0}^{n(T)-1}(\Delta w_{t_i})^3(\Delta w_{t_j})^3\right) = 2E\sum_{k=0}^{n(T)-1}(\Delta w_{t_k})^6$$ by independence b/t Δw_{t_i} and Δw_{t_j}, $\forall i\neq j$

$$= 2\sum_{k=0}^{n(T)-1} E(\Delta w_{t_k})^6$$

$$= \sum_{k=0}^{n(T)-1} 30(\Delta t_k)^3 \quad \text{as } \forall k.\ \Delta w_{t_k}\sim N(0,\Delta t_k)$$

$$\leq 30\max_k(\Delta t_k)^2\sum_{k=0}^{n(T)-1}\Delta t_k = 30T\max_k(\Delta t_k)^2 \longrightarrow 0 \quad \text{as } \max_k \Delta t_k\to 0.$$

$$② \xrightarrow{L^2} 0$$

$$E\left(③-3\sum_{k=0}^{n(T)-1} w_{t_k}\Delta t_k\right)^2 = 9E\left(\sum_{i=0}^{n(T)-1}\sum_{j=0}^{n(T)-1} w_{t_i}w_{t_j}(\Delta w_{t_i}^2-\Delta t_i)(\Delta w_{t_j}^2-\Delta t_j)\right)$$

$$= 9E\left(\sum_{k=1}^{n(T)-1} w_{t_k}^2(\Delta w_{t_k}^2-\Delta t_k)^2\right)$$ by independence b/t $w_{t_i}w_{t_j}(\Delta w_{t_i}^2-\Delta t_i)$ and $(\Delta t_j^2-\Delta t_j)$ for $\forall i<j$ as $(\Delta w_{t_i}^2-\Delta t_i)_{i=0}^{n(T)-1}$ is a mtg

$$= 9\sum_{k=1}^{n(T)-1} E\left(w_{t_k}^2 E\left((\Delta w_{t_k}^2-\Delta t_k)^2 \mid \mathcal{F}_{t_k}\right)\right)$$

$$= 9\sum_{k=1}^{n(T)-1} E\left(w_{t_k}^2\, 2(\Delta t_k)^2\right) \quad \text{as } E\left((\Delta w_{t_k}^2-\Delta t_k)^2\mid\mathcal{F}_{t_k}\right) = E\left(\Delta w_{t_k}^4-2\Delta t_k\Delta w_{t_k}^2+\Delta t_k^2\mid\mathcal{F}_{t_k}\right)$$

$$= 3\Delta t_k^2 - 2\Delta t_k\cdot\Delta t_k + \Delta t_k^2 = 2\Delta t_k^2$$

$$\leq 18T\left(\max_k \Delta t_k\right)\sum_{k=1}^{n(T)-1}\Delta t_k = 18T^2\max_k\Delta t_k \longrightarrow 0 \quad \text{as } \max_k \Delta t_k\to 0$$

$$③ \xrightarrow{L^2} \lim_{\max_k \Delta t_k\to 0} 3\sum_{k=0}^{n(T)-1} w_{t_k}\Delta t_k = 3\int_0^T w_t\,dt$$

$$\overset{①②\&③}{\Rightarrow} \int_0^T w^2(t)\,dw(t) = \frac{1}{3}w_T^3 - \int_0^T w_t\,dt.$$

#11. Given $\int_0^T X_t \circ dW_t := \lim_{\max_k (t_{k+1}-t_k)\to 0} \sum_{k=0}^{n(T)-1} \frac{X_{t_{k+1}} + X_{t_k}}{2} (W_{t_{k+1}} - W_{t_k})$,

WTS: ① $\int_0^T X_t \circ dW_t - \int_0^T X_t \, dW_t$ in terms of $\langle X, W \rangle_T := \lim_{\max_k \Delta t_k \to 0} \sum_{k=0}^{n(T)-1} (X_{t_{k+1}} - X_{t_k})(W_{t_{k+1}} - W_{t_k})$

② $\int_0^T W_t \circ dW_t$

①. $\int_0^T X_t \circ dW_t - \int_0^T X_t \, dW_t$

$$= \lim_{\max_k (t_{k+1}-t_k)\to 0} \sum_{k=0}^{n(T)-1} \frac{X_{t_{k+1}} + X_{t_k} - 2X_{t_k}}{2} (W_{t_{k+1}} - W_{t_k}) \qquad \text{limit in } L^2(\Omega, \mathcal{F}, P).$$

$$= \lim_{\max_k (t_{k+1}-t_k)\to 0} \frac{1}{2} \sum_{k=0}^{n(T)-1} (X_{t_{k+1}} - X_{t_k})(W_{t_{k+1}} - W_{t_k}) = \frac{1}{2} \langle X, W \rangle_T \quad \text{by definition.}$$

②. $\int_0^T W_t \circ dW_t$

$$= \lim_{\max_k (t_{k+1}-t_k)\to 0} \sum_{k=0}^{n(T)-1} (W_{t_{k+1}} + W_{t_k})(W_{t_{k+1}} - W_{t_k}) \frac{1}{2}$$

$$= \lim_{\max_k (t_{k+1}-t_k)\to 0} \sum_{k=0}^{n(T)-1} (W_{t_{k+1}}^2 - W_{t_k}^2) \frac{1}{2}$$

$$= \lim_{\max_k (t_{k+1}-t_k)\to 0} (W_T^2 - 0) \frac{1}{2} = \frac{1}{2} W_T^2$$

$$\text{if } s<t, \qquad = E(M_t^2 - M_s^2 \mid \mathcal{F}_s)$$

#12. Given $(M_t, \mathcal{F}_t) \in \mathcal{M}_2^c$, WTS: $E[(M_t - M_s)^2 \mid \mathcal{F}_s] = E[\langle M\rangle_t - \langle M\rangle_s \mid \mathcal{F}_s]$

Proof: $E[(M_t - M_s)^2 \mid \mathcal{F}_s]$

$$= E(M_t^2 \mid \mathcal{F}_s) - 2E(M_t M_s \mid \mathcal{F}_s) + E(M_s^2 \mid \mathcal{F}_s)$$

$$\begin{aligned} E(M_t M_s \mid \mathcal{F}_s) &= E[(M_t - M_s)M_s \mid \mathcal{F}_s] + E(M_s^2 \mid \mathcal{F}_s) \\ &= M_s E(M_t - M_s \mid \mathcal{F}_s) + E(M_s^2 \mid \mathcal{F}_s) \\ &= E(M_s^2 \mid \mathcal{F}_s) \end{aligned}$$

$$(*) \Rightarrow E[(M_t - M_s)^2 \mid \mathcal{F}_s] = E(M_t^2 - 2M_s^2 + M_s^2 \mid \mathcal{F}_s) = E(M_t^2 - M_s^2 \mid \mathcal{F}_s)$$

Now, $M \in \mathcal{M}_2^c \Rightarrow (M_t^2, \mathcal{F}_t)$ is a continuous submartingale.

Claim $(M_t^2 - \langle M\rangle_t, \mathcal{F}_t) \in \mathcal{M}^c$.

Indeed, $E[(M_t^2 - \langle M\rangle_t) - (M_s^2 - \langle M\rangle_s) \mid \mathcal{F}_s]$

$$\begin{aligned} &= E[(M_t^2 - M_s^2) - (\langle M\rangle_t - \langle M\rangle_s) \mid \mathcal{F}_s] \\ &= E\Big[(M_t^2 - M_s^2) - \lim_{\max_k \Delta t_k \to 0} \sum_{k=\tilde n(s)}^{n(t)-1} (M_{t_{k+1}} - M_{t_k})^2 \;\Big|\; \mathcal{F}_s\Big] \\ &= \lim_{\max_k \Delta t_k \to 0} \sum_{k=\tilde n(s)}^{n(t)-1} E[(M_{t_{k+1}}^2 - M_{t_k}^2) \mid \mathcal{F}_s] - E[(M_{t_{k+1}} - M_{t_k})^2 \mid \mathcal{F}_s] \quad \text{by bdd convergence as } M \in \mathcal{M}_2^c \\ &= \lim_{\max_k \Delta t_k \to 0} \sum_{k=\tilde n(s)}^{n(t)-1} E[(M_{t_{k+1}}^2 - M_{t_k}^2) - (M_{t_{k+1}}^2 - M_{t_k}^2) \mid \mathcal{F}_s] \quad \text{by } (*) . \\ &= 0, \quad \text{if } s<t. \end{aligned}$$

$$\begin{aligned} \Rightarrow E(M_t^2 - M_s^2 \mid \mathcal{F}_s) &= E[(M_t^2 - \langle M\rangle_t) - (M_s^2 - \langle M\rangle_s) \mid \mathcal{F}_s] + E(\langle M\rangle_t - \langle M\rangle_s \mid \mathcal{F}_s) \\ &= E(\langle M\rangle_t - \langle M\rangle_s \mid \mathcal{F}_s) \quad \text{as } (M_t^2 - \langle M\rangle_t, \mathcal{F}_t) \in \mathcal{M}^c \end{aligned}$$

$\square$

#13. Given $(M_t, \mathcal{F}_t) \in \mathcal{M}_2^c$, $X = \sum_{i=0}^{n} \xi_i \mathbb{1}_{(t_i,\, t_i + \Delta t_i]}$ is simple, $\Delta t_i := t_{i+1} - t_i$

WTS: $E[((X\cdot M)_t - (X\cdot M)_s)^2 \mid \mathcal{F}_s] = E[\int_s^t X_r^2 \, d\langle M\rangle_r \mid \mathcal{F}_s]$, $s<t$

Proof: $E[((X\cdot M)_t - (X\cdot M)_s)^2 \mid \mathcal{F}_s]$

$= E[(\int_s^t X_r \, dM_r)^2 \mid \mathcal{F}_s] = E[\left(\sum_{k=\tilde{n}(s)}^{n(t)-1} \xi_k (M_{t_{k+1}} - M_{t_k})\right)^2 \mid \mathcal{F}_s]$

where $X\mathbb{1}_{[s,t]} = \xi_{\tilde{n}(s)-1} \mathbb{1}_{\{t_{\tilde{n}(s)}\}} + \sum_{k=\tilde{n}(s)}^{n(t)-1} \xi_k \mathbb{1}_{(t_k,\, t_k+\Delta t_k]}$

$= E[\sum_{i=\tilde{n}(s)}^{n(t)-1} \sum_{j=\tilde{n}(s)}^{n(t)-1} \xi_i (M_{t_{i+1}} - M_{t_i}) \xi_j (M_{t_{j+1}} - M_{t_j}) \mid \mathcal{F}_s]$

$\Delta M_{t_j} = M_{t_{j+1}} - M_{t_j}$, $\forall j$

$= \underbrace{E[\sum_{k=\tilde{n}(s)}^{n(t)-1} \xi_k^2 (M_{t_{k+1}} - M_{t_k})^2 \mid \mathcal{F}_s]}_{①} + \underbrace{2\sum_{i<j} E[\xi_i \xi_j \Delta M_{t_i} \Delta M_{t_j} \mid \mathcal{F}_s]}_{②}$

$② = 2 \sum_{i<j} E[\xi_i \xi_j \Delta M_{t_i} E(\Delta M_{t_j} \mid \mathcal{F}_{t_j}) \mid \mathcal{F}_s]$ by tower property

$= 2 \sum_{i<j} E[\xi_i \xi_j \Delta M_{t_i} (M_{t_j} - M_{t_j}) \mid \mathcal{F}_s]$ as $(M_t, \mathcal{F}_t) \in \mathcal{M}$

$= 0$

$① = \sum_{k=\tilde{n}(s)}^{n(t)-1} E[\xi_k^2 (M_{t_{k+1}} - M_{t_k})^2 \mid \mathcal{F}_s]$

$= \sum_{k=\tilde{n}(s)}^{n(t)-1} E[\xi_k^2 E((M_{t_{k+1}} - M_{t_k})^2 \mid \mathcal{F}_{t_k}) \mid \mathcal{F}_s]$ by tower property

$= \sum_{k=\tilde{n}(s)}^{n(t)-1} E[\xi_k^2 (\langle M\rangle_{t_{k+1}} - \langle M\rangle_{t_k}) \mid \mathcal{F}_s]$ by #12 (HW#3)

$= E[\sum_{k=\tilde{n}(s)}^{n(t)-1} \xi_k^2 (\langle M\rangle_{t_{k+1}} - \langle M\rangle_{t_k}) \mid \mathcal{F}_s]$

Since $X^2 \mathbb{1}_{[s,t]} = \xi_{\tilde{n}(s)-1}^2 \mathbb{1}_{\{t_{\tilde{n}(s)}\}} + \sum_{k=\tilde{n}(s)}^{n(t)-1} \xi_k^2 \mathbb{1}_{(t_k,\, t_k+\Delta t_k]}$, we have:

$\int_s^T X_r^2 \, d\langle M\rangle_r = \sum_{k=\tilde{n}(s)}^{n(t)-1} \xi_k^2 (\langle M\rangle_{t_{k+1}} - \langle M\rangle_{t_k})$

$\Rightarrow ① = E[\int_s^T X_r^2 \, d\langle M\rangle_r \mid \mathcal{F}_s]$

$\overset{① \& ②}{\Rightarrow} E[((X\cdot M)_t - (X\cdot M)_s)^2 \mid \mathcal{F}_s] = E[\int_s^T X_r^2 \, d\langle M\rangle_r \mid \mathcal{F}_s]$ □

#14. Given $M \in \mathcal{M}_2^c$, WTS: $Y\cdot(X\cdot M) = (YX)\cdot M$ for X, Y simple

Also, find $\overset{\text{weaker}}{\widehat{\ }}$ condition for X, Y s.t. $Y\cdot(X\cdot M) = (YX)\cdot M$ to hold

Proof: Y simple $\Rightarrow$ $Y = \sum_{j=0}^{n(T)-1} \xi_j^Y \mathbb{1}_{(t_j, t_{j+1}]} + \xi_0^Y \mathbb{1}_{\{t_0\}}$, $\forall\, T \geq 0$.

X simple $\Rightarrow$ $X = \sum_{j=0}^{n(T)-1} \sum_{i=0}^{n(t_j)-1} \xi_{j_i}^X \mathbb{1}_{(t_{j_i}, t_{j_{i+1}}]} + \xi_0^X \mathbb{1}_{\{t_0\}}$, ξ_i^X's not necessarily distinct by design

Note: $XY = \sum_{j=0}^{n(T)-1} \xi_j^Y \sum_{i=1}^{n(t_j)-1} \xi_{j_i}^X \mathbb{1}_{(t_{j_i}, t_{j_{i+1}}]} + \mathbb{1}_{\{t_0\}} \xi_0^X \xi_0^Y$ (*)

Now, $(X\cdot M)_T = \sum_{j=0}^{n(T)-1} \sum_{i=0}^{n(t_j)-1} \xi_{j_i}^X (M_{t_{j_{i+1}}} - M_{t_{j_i}})$

$$
\begin{aligned}
Y\cdot(X\cdot M) &= \sum_{j=0}^{n(T)-1} \xi_j^Y \left((X\cdot M)_{t_{j+1}} - (X\cdot M)_{t_j}\right) \\
&= \sum_{j=0}^{n(T)-1} \xi_j^Y \sum_{i=0}^{n(t_j)-1} \xi_{j_i}^X (M_{t_{j_{i+1}}} - M_{t_{j_i}}) \\
&= \sum_{j=0}^{n(T)-1} \sum_{i=0}^{n(t_j)-1} \xi_j^Y \xi_{j_i}^X (M_{t_{j_{i+1}}} - M_{t_{j_i}}) \\
&= \int_0^T (XY)_s \, dM_s = (XY)\cdot M
\end{aligned}
$$

$\square$

#15. Given $(M_t, \mathcal{F}_t) \in \mathcal{M}_2^c$, X, Y bdd, WTS: $\langle X\cdot M, Y\cdot M\rangle_t = \int_0^t X_s Y_s \, d\langle M\rangle_s$

Proof: ① We first show $\langle X\cdot M, Y\cdot M\rangle_t = \int_0^t X_s Y_s \, d\langle M\rangle_s$ for X, Y simple

② Then, we approximate bdd process simple processes to finish the proof.

①. X simple $\Rightarrow$ $X = \sum_{j=0}^{n(T)-1} \xi_j^X \mathbb{1}_{(t_j, t_{j+1}]} + \xi_0^X \mathbb{1}_{\{0\}} = \sum_{j=0}^{n(T)-1} \sum_{i=0}^{n(t_{j+1})-1} \xi_{ji}^X \mathbb{1}_{(t_{ji}, t_{ji+1}]} + \xi_0^X \mathbb{1}_{\{0\}}$

Y simple $\Rightarrow$ $Y = \sum_{j=0}^{n(T)-1} \sum_{i=0}^{n(t_{j+1})-1} \xi_{ji}^Y \mathbb{1}_{(t_{ji}, t_{ji+1}]} + \xi_0^Y \mathbb{1}_{\{0\}}$

Now, $\langle X\cdot M, Y\cdot M\rangle_t = \frac{1}{4}[\langle X\cdot M + Y\cdot M\rangle_t - \langle X\cdot M - Y\cdot M\rangle_t]$.

Since $E[(\int_0^t (X_s+Y_s)\, dM_s - \int_0^s (X_s+Y_s)\, dM_s)^2 \mid \mathcal{F}_s] = E[\int_s^t (X+Y)_r^2 \, d\langle M\rangle_r \mid \mathcal{F}_s]$

by #13 (HW#3), we have $\langle X\cdot M + Y\cdot M\rangle_t = \int_0^t (X+Y)_r^2 \, d\langle M\rangle_r$ by definition.

Similarly, we have $\langle X\cdot M - Y\cdot M\rangle_t = \int_0^t (X-Y)_r^2 \, d\langle M\rangle_r$.

$$\Rightarrow \langle X\cdot M, Y\cdot M\rangle_t = \frac{1}{4}\left[\int_0^t (X+Y)_r^2 \, d\langle M\rangle_r - \int_0^t (X-Y)_r^2 \, d\langle M\rangle_r\right.$$

$$= \frac{1}{4}\left[\sum_{j=0}^{n(t)-1} \sum_{i=0}^{n(t_{j+1})-1} [(\xi_{ji}^X + \xi_{ji}^Y)^2 - (\xi_{ji}^X - \xi_{ji}^Y)^2]\right] (\langle M\rangle_{t_{ji+1}} - \langle M\rangle_{t_{ji}})$$

$$= \sum_{j=0}^{n(t)-1} \sum_{i=0}^{n(t_{j+1})-1} \xi_{ji}^X \xi_{ji}^Y (\langle M\rangle_{t_{ji+1}} - \langle M\rangle_{t_{ji}})$$

$$= \int_0^T X_s Y_s \, d\langle M\rangle_s \quad \text{by definition}$$

□

#16. Given X by $X_t := e^{\lambda t} X_0 + \varepsilon e^{\lambda t} \int_0^t e^{-\lambda s} dW_s$, $t \geq 0, \lambda \in \mathbb{R}, \varepsilon > 0$

WTS: $dX_t = \lambda X_t dt + \varepsilon dW_t$.

Proof:
$$X_t = e^{\lambda t} X_0 + \varepsilon e^{\lambda t} \int_0^t e^{-\lambda s} dW_s$$
$$e^{-\lambda t} X_t = X_0 + \varepsilon \int_0^t e^{-\lambda s} dW_s$$
$$\Rightarrow \quad d(e^{-\lambda t} X_t) = \varepsilon e^{-\lambda t} dW_t$$

Now, by Itô's formula, we have $g(t,x) = e^{-\lambda t} x \in C^{1,2}$ implies
$$dg(t,X_t) = \frac{\partial}{\partial t} g(t,X_t)\, dt + \frac{\partial}{\partial x} g(t,X_t)\, dX_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} g(t,X_t) (dX_t)^2$$
$$= (-\lambda) e^{-\lambda t} X_t dt + e^{-\lambda t} dX_t + 0$$
$$\Rightarrow \quad \varepsilon e^{-\lambda t} dW_t = (-\lambda) e^{-\lambda t} X_t\, dt + e^{-\lambda t} dX_t$$
$$dX_t = \lambda X_t\, dt + \varepsilon\, dW_t.$$

$\square$

#17. Given $f:[0,+\infty)\to\mathbb{R}$ deterministic, bdd in $[0,t]$, $\forall t$,

WTS: ①. $X_t := \int_0^t f(s)\,dW_s \Rightarrow (X_t, \mathcal{F}_t^W)$ is a Gaussian process.

②. $E(X_t)$ and $E(X_t X_s)$

①. Proof: Since $f(s)$ is bdd, $\exists \{f^n\}_{n\in\mathbb{N}}$, a sequence of simple functions, s.t.

$f^n \xrightarrow[n\to\infty]{} f$ in L^2, as simple functions dense in $L^2([0,+\infty),\mathcal{B}[0,+\infty),\text{Leb})$.

Now, for each $n\in\mathbb{N}$, define (X_t^n) by $X_t^n := \int_0^t f^n(s)\,dW_s$,

Pick $\{t_i\}_{i=1}^{\hat{n}} \subset [0,+\infty)$ arbitrarily, we have, $\forall i\in\{1,\dots,\hat{n}\}$.

$$E\left(X^n_{t_{i+1}-t_i} - X_{t_{i+1}-t_i}\right)^2 = E\left(\int_{t_i}^{t_{i+1}} f^n(s)\,dW_s - \int_0^{t_{i+1}} f(s)\,dW_s\right)^2$$

$$= E\left(\int_{t_i}^{t_{i+1}} (f^n - f)^2\, d\langle W\rangle_s\right) \quad \text{by } (*) \text{ below}$$

$$= \int_{t_i}^{t_{i+1}} (f^n - f)^2\, ds \le \|f^n - f\|_{L^2}^2 \xrightarrow[n\to\infty]{} 0$$

$(*)$ If $X\in\mathcal{L}_0$, $E(I_t^W(X)^2) = E\left(\sum_{j=0}^{n-1} X_{t_j}(W_{t_{j+1}} - W_{t_j})\right)^2 = E\sum_{j=0}^{n-1} X_{t_j}^2 (W_{t_{j+1}} - W_{t_j})^2$

$$= \sum_{j=0}^{n-1} E\left(X_{t_j}^2\, E[(X_{t_{j+1}} - X_{t_j})^2 \mid \mathcal{F}_{t_j}]\right)$$

$$= \sum_{j=0}^{n-1} E[X_{t_j}^2(\langle X\rangle_{t_{j+1}} - \langle X\rangle_{t_j})] = E\int_0^t X_s^2\, d\langle X\rangle_s$$

If $X\in\mathcal{L}^*$, since $\mathcal{L}_0$ is dense in $\mathcal{L}^*$ w.r.t. $[\cdot]$, $\exists \{X^n\}_{n\in\mathbb{N}} \subset \mathcal{L}_0$

s.t. $[X^n - X] \xrightarrow[n\to\infty]{} 0$, then $E(I_t^W(X))^2 = \lim_{n\to\infty} E(I_t^W(X^n))^2$

$$= \lim_{n\to\infty} E\left(\int_0^t (X^n)^2\, d\langle W\rangle_s\right) = E\left(\int_0^t X^2\, d\langle W\rangle_s\right)$$

Since $E\int_0^t (f_s^n - f_s)^2\, ds \le Kt < \infty$, progressive measurability and adaptedness follow trivially from deterministic nature. $f^n - f\in\mathcal{L}^*$.

$$\Rightarrow E\left(I^W_{t_{i+1}-t_i}(f^n - f)\right)^2 = E\int_{t_i}^{t_{i+1}} (f^n - f)^2\, d\langle W\rangle_s$$

Now, $X^n_{t_{i+1}-t_i} \xrightarrow[n\to\infty]{L^2} X_{t_{i+1}-t_i} \Rightarrow X^n_{t_{i+1}-t_i} \xrightarrow[n\to\infty]{D} X_{t_{i+1}-t_i}$ (in distribution)

But $X_{t_{i+1}-t_i} = \int_{t_i}^{t_{i+1}} f^n(s) dw_s = \sum_{k^i=n(t_i)}^{n(t_{i+1})-1} c_{k^i}^i (w_{k^i+1} - w_{k^i})$, $c_{k^i} \in \mathbb{R}$, $\forall k^i \in \{0, \cdots, n(t_{i+1})-1\}$

and $\{w_{k^i+1} - w_{k^i}\}_{k^i, i}$ is a Gaussian vector w/ independent components,

$\forall i$, $X^n_{t_{i+1}-t_i} = \sum_{k^i=n(t_i)}^{n(t_{i+1})-1} c_{k^i}(w_{k^i+1} - w_{k^i})$ is Gaussian

$\Rightarrow$ X_{t_i} is Gaussian as $X^n_{t_i} \xrightarrow{D} X_{t_i}$, $\forall i \in \{1, \cdots, \hat{n}\}$.

Since $\{X^n_{t_{i+1}-t_i}\}_{i=1}^{\hat{n}} = \left\{\sum_{k^i=n(t_i)}^{n(t_{i+1})-1} (w_{k^i+1} - w_{k^i})\right\}_{i=1}^{\hat{n}}$ is also Gaussian vector w/ independent components, $\{X_{t_{i+1}-t_i}\}_{i=1}^{\hat{n}}$ is Gaussian vector w/ independent components as $\{X^n_{t_{i+1}-t_i}\}_{i=1}^{\hat{n}} \xrightarrow[n\to\infty]{D} \{X_{t_{i+1}-t_i}\}_{i=1}^{\hat{n}}$.

Since $\{X_{t_i}\}_{i=1}^{\hat{n}} = A \{X_{t_{i+1}-t_i}\}_{i=1}^{\hat{n}}$, $A = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{\hat{n}\times\hat{n}}$,

$\{X_{t_i}\}_{i=1}^{\hat{n}}$ is a Gaussian vector.

Finally, since the choice of $\{t_i\}_{i=1}^{\hat{n}} \subset [0, +\infty)$ is arbitrary, we have $\{X_t\}_{t\in[0,\infty)}$ is a Gaussian process.

$\square$

②. By similar argument on (*), we have $f \in \mathcal{L}^*(M)$, $w \in \mathcal{M}_2^c \Rightarrow I_t^w(f) \in \mathcal{M}_2^c$

$$\Rightarrow E(X_t) = E(I_t^w(f)) = E(I_0^w(f)) = 0$$

$$E(X_t X_s) \underset{\substack{s\le t \\ \text{WLOG}}}{=} E(I_t^w(f) I_s^w(f)) = E[E(I_{t-s}^w(f) | \mathcal{F}_s) I_s^w(f)] + E(I_s^w(f))^2$$

$$= E \int_0^s f^2(r) d\langle w\rangle_r = \int_0^s f^2(r) dr$$

$$\Rightarrow C(t,s) = \int_0^{t\wedge s} (f(r))^2 dr$$

#18. WT: Find all the values of λ in #16 s.t. $\exists\, a, \sigma^2$ s.t. $X_0 \sim N(a, \sigma^2)$, then (X_t) is a stationary process.

Proof: ① Let $\lambda \neq 0$, then $e^{-\lambda t} X_t = X_0 + \varepsilon \int_0^t e^{-\lambda s} dW_s$, $\forall\, t \in [0, +\infty)$.

Since $e^{-\lambda s}$ is bdd $Y_t := \int_0^t e^{-\lambda s} dW_s$ is Gaussian w/ $E(Y_t) = 0$

$E(Y_t Y_s) = \int_0^{t \wedge s} e^{-2\lambda r} dr = \frac{-1}{2\lambda}\left(e^{-2\lambda(t\wedge s)} - 1\right) = \frac{1}{2\lambda}\left(1 - e^{-2\lambda(t\wedge s)}\right)$

by #17 HW#3.

Since $X_0 \sim N(a, \sigma^2)$, $Y_t \sim N\left(0, \frac{1}{2\lambda}(1 - e^{-2\lambda(t\wedge s)})\right)$, and $\forall\, t \in [0, +\infty)$

$X_0 \perp W_t \Rightarrow X_0 \perp Y_t$, we have:

$E(X_t) = e^{\lambda t} E(e^{-\lambda t} X_t) = e^{\lambda t}(a + 0) = e^{\lambda t} a$.

$$\begin{aligned} E(X_t X_s) &= e^{\lambda(t+s)} E\left(e^{-\lambda t} X_t\, e^{-\lambda s} X_s\right) \\ &= e^{\lambda(t+s)} E\left(X_0^2 + \varepsilon X_0 Y_t + \varepsilon X_0 Y_s + \varepsilon^2 Y_t Y_s\right) \\ &= e^{\lambda(t+s)}\left(\sigma^2 + 0 + 0 + \frac{\varepsilon^2}{2\lambda}(1 - e^{-2\lambda(t\wedge s)})\right) \\ &= e^{\lambda(t+s)}\sigma^2 + \frac{\varepsilon^2}{2\lambda} e^{\lambda(t+s)} - \frac{\varepsilon^2}{2\lambda} e^{\lambda(t-s)} \end{aligned}$$

Now, X_t is Gaussian $\Rightarrow$ X_t is stationary $\Leftrightarrow$ it's weakly stationary

X_t is weak stationary $\Leftrightarrow \begin{cases} e^{\lambda t} a \equiv \text{constant} \Leftrightarrow a = 0 \text{ as } \lambda \neq 0. \\ (\sigma^2 + \frac{\varepsilon^2}{2\lambda}) e^{\lambda(t+s)} - \frac{\varepsilon^2}{2\lambda} e^{\lambda(t-s)} = f(t-s) \end{cases}$

$\Leftrightarrow \sigma^2 + \frac{\varepsilon^2}{2\lambda} = 0$, $\lambda = -(\frac{\varepsilon}{\sigma})^2 < 0$

Therefore, we have if $\lambda \neq 0$, we must have

$\lambda < 0$, then $\exists \begin{cases} a = 0 \\ \sigma = (-\frac{\varepsilon^2}{2\lambda})^{\frac{1}{2}} \end{cases}$ guarantees (X_t) be stationary.

②. Let $\lambda = 0$, then we have:

$$X_t = X_0 + \varepsilon \int_0^t dW_s = X_0 + \varepsilon W_t \sim \mathcal{N}(a, \sigma^2 + t) \quad \text{as } X_0 \perp W_t.$$

Since X_t is Gaussian, X_t stationary $\Leftrightarrow$ it's weakly so.

X_t is weakly stationary $\Leftrightarrow \begin{cases} E(X_t) \equiv \text{constant} \\ E(X_t X_s) = f(t-s) \end{cases}$

$E(X_t) \equiv a$ is constant

$$\begin{aligned} E(X_t X_s) &= E\left(X_0^2 + \varepsilon X_0 W_t + \varepsilon X_0 W_s + W_t W_s \right) \\ &= \sigma^2 + s \wedge t + E\left[W_{t\wedge s} E\left(W_{t \vee s} - W_{t\wedge s} \mid \mathcal{F}_{t\wedge s} \right) \right] \\ &= \sigma^2 + s \wedge t \end{aligned}$$

$\Rightarrow$ $E(X_t X_s)$ cannot be a function of $s - t$ for any σ^2

$\Rightarrow$ X_t cannot be stationary when $\lambda = 0$.

By ①, ②, we show only when $\lambda < 0$, $\exists \begin{cases} a = 0 \\ \sigma = \left(-\frac{\varepsilon^2}{2\lambda}\right)^{\frac{1}{2}} \end{cases}$ to guarantee the stationarity of X_t

#19. (Feynman - Kac Formula)

WTS: $E\, e^{\int_0^t \phi(t-s,x+W_s)ds} u_0(x+W_t)$ is a solution to the Cauchy problem:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_{xx} u(t,x) + \phi(t,x)u(t,x), & (*) \\ u(0,x) = u_0(x). \end{cases}$$

Let $Y_s := e^{-R(s)} u(t-s, W_s+x)$ on $[0,t]$ for $\begin{cases}\text{fixed } t\in[0,+\infty). \\ \text{fixed } x\in\mathbb{R}\end{cases}$

$R(s) := \int_0^s \phi(t-r, x+W_r)dr$

Since $\begin{cases} u \text{ is smooth on } [0,+\infty)\times\mathbb{R} \text{ by assumption}, & u\in C^{1,2}_{bdd}([0,t]\times\mathbb{R}). \\ \phi \text{ is smooth and bdd on } [0,+\infty)\times\mathbb{R}, & \phi\in C^{1,2}_{bdd}([0,t],\mathbb{R}) \end{cases}$, we have:

$f := e^{-\int_0^s \phi(t-r,x+W_r)dr} u(t-s, W_s+x) \in C^{1,2}_{bdd}([0,t],\mathbb{R})$.

Since $W\in\mathcal{M}^{c,loc}$, we can apply Itô's formula:

$$\begin{aligned} dY_s &= \phi(t-s,x+W_s)e^{R(s)}u(t-s,W_s+x)ds - e^{R(s)}u_t(t-s,W_s+x)ds \\ &\quad + e^{R(s)}u_x(t-s,W_s+x)dW_s + e^{R(s)}\tfrac{1}{2}u_{xx}(t-s,W_s+x)d\langle W\rangle_s \\ &= e^{R(s)}u_x(t-s,W_s+x)dW_s + [\phi u - u_t + \tfrac{1}{2}u_{xx}]e^{R(s)}ds \\ &= e^{R(s)}u_x(t-s,W_s+x)dW_s + 0 \qquad \text{by } (*) \end{aligned}$$

$\Rightarrow Y_s = \int_0^s e^{R(r)}u_x(t-r,W_r+x)dW_r$

Since $e^{R(r)}u_x(t-r,W_r+x) \in C_{bdd}([0,t],\mathbb{R})$, $\int_0^t (e^{R(r)}u_x(t-r,W_r+x))^2 d\langle W\rangle_r < \infty$

$\Rightarrow e^{R(r)}u_x(t-r,W_r+x)|_{[0,t]} \in \mathcal{L}^*(W)$

$\Rightarrow Y_s|_{[0,t]} \in \mathcal{M}_2^c$

Now, Y_s is a mtg. on $[0,t]$ implies $Y_0 = E\,Y_t$

But $\begin{cases} Y_0 = e^{R(0)}u(t,W_0+x) = u(t,x) \\ E\,Y_t = E\,e^{R(t)}u(0,W_t+x) = E[e^{\int_0^t \phi(t-r,x+W_r)dr} u_0(x+W_t)] \end{cases}$

$\square$

#20. Given D is open bdd w/ smooth ∂D, $g: D \to \mathbb{R}$, $f: \partial D \to \mathbb{R}$ cts., and $u \in C(\bar{D}) \cap C^2(D)$ satisfying $\begin{cases} \frac{1}{2}\Delta u(x) = -g(x), & x \in D \\ u(x) = f(x), & x \in \partial D. \end{cases}$

WTS: $u(x) = E\left[f(x+W_\tau) + \int_0^\tau g(x+W_t)\,dt\right]$, $\tau := \inf\{t \geq 0;\ x+W_t \in \partial D\}$

Proof: Since D is bdd, we have for $\forall\ x \in D$, $P\{\tau < \infty\} = 1$ as $x + W_t$ is unbounded $\forall\ x \in D$.

Since $u \in C^2(D)$, define $\begin{cases} D_n := \{x \in D;\ \inf_{y \in \partial D} \|x-y\| > \frac{1}{n}\}, \text{ by Itô}: \\ B_n := \{x \in \mathbb{R}^d:\ \|x\| < n\} \end{cases}$

$u(x+W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) = u(x) + \sum_{i=1}^d \int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} \frac{\partial u}{\partial x_i}(x+W_s)\,dW_s^{(i)} + \int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} -g(x+W_s)\,ds$

Since $\frac{\partial u}{\partial x_i} \in C^1(\overline{B_n \cap D_n})$ is bdd, $\sum_{i=1}^d \int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} \frac{\partial u}{\partial x_i}(x+W_s)\,dW_s^{(i)} =: M_{t\wedge\tau_{B_n}\wedge\tau_{D_n}} \in \mathcal{M}_2^c$

$\Rightarrow E(M_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) = M_0 = 0$

$\Rightarrow u(x) = E\, u(x+W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}}) + E\int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} g(x+W_s)\,ds$

Now let $n \to \infty$, $t \to \infty$, we have:

$$u(x) = \underbrace{\lim_{\substack{t\to\infty \\ n\to\infty}} E\, u(x+W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}})}_{①} + \underbrace{\lim_{\substack{t\to\infty \\ n\to\infty}} E\int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} g(x+W_s)\,ds}_{②}$$

② $= E \lim_{\substack{t\to\infty \\ n\to\infty}} \int_0^{t\wedge\tau_{B_n}\wedge\tau_{D_n}} g(x+W_s)\,ds = E\int_0^\tau g(x+W_s)\,ds$ by bounded convergence thm as $\int_0^\tau g(x+W_s)\,ds \leq K\, V_D$ where $g \leq K$ on $\bar{D}$ by continuity

① $= E \lim_{\substack{t\to\infty \\ n\to\infty}} u(x+W_{t\wedge\tau_{B_n}\wedge\tau_{D_n}})$ by bdd convergence

$= E\, f(x+W_\tau)$ as $u \in C^1(\bar{D}) \Rightarrow \lim_{\substack{x\to a\in\partial D \\ x\in D}} u(x) = f(a)$.

By ① & ②, $u(x) = E\, f(x+W_\tau) + E\int_0^\tau g(x+W_s)\,ds$, $\forall\ x \in D$

continue next page

Now, since ∂D is smooth by assumption, $\forall a \in \partial D$, a is regular for D.

Then, $f : \partial D \to \mathbb{R}$ is cts. $\Rightarrow \lim_{\substack{x \to a \\ x \in D}} E f(x + W_\tau) = f(a)$, $\forall a \in \partial D$.

$\Rightarrow$ $\forall x \in \partial D$, let $\{x_n\}_n \subset D$ s.t. $x_n \xrightarrow[n\to\infty]{} x$, we have:

$$u(x) = \underbrace{\lim_{n\to\infty} E f(x_n + W_\tau)}_{①} + \underbrace{\lim_{n\to\infty} \int_0^\tau g(x_n + W_s)\, ds}_{②}.$$

① $= f(x)$ by regularity of x for ∂D

② $= 0$ as x is regular for ∂D $\Rightarrow$ $\tau = 0$ P-a.s. $\forall x \in \partial D$.

$\Rightarrow \int_0^0 g(x + W_s)\, ds = 0$.

By ①, ②, $u(x) = f(x)$, $\forall x \in \partial D$, as desired.

Clearly, $u(x) = E f(x + W_\tau) + E \int_0^\tau g(x + W_s)\, ds$ satisfies $-\Delta u = 0$ in D.

$\Rightarrow$ $u(x)$ is a sol to (D, f).

$\square$

#21.① Suppose $a \in \mathbb{R}$, $\sigma > 0$, $x_0 > 0$, show $dS_t = A S_t dt + B S_t dW_t$ for some $A, B \in \mathbb{R}$ given $S_t := x_0 e^{at + \sigma W_t}$

Let $g(t, w) = x_0 e^{at + \sigma w}$, then by Itô's formula, we have

$$dg(t, W_t) = \frac{\partial}{\partial t} g(t, W_t)\, dt + \frac{\partial}{\partial w} g(t, W_t)\, dW_t + \frac{1}{2} \frac{\partial^2}{\partial w^2} g(t, W_t) (dW_t)^2$$

$$= x_0 e^{at + \sigma W_t} \left[a\,dt + \sigma\, dW_t + \frac{\sigma^2}{2} (dW_t)^2 \right]$$

$$= g(t, W_t) \left[(a + \frac{\sigma^2}{2}) dt + \sigma\, dW_t \right]$$

Now, since $S_t = g(t, W_t)$, we have

$$dS_t = S_t \left[(a + \frac{\sigma^2}{2})\, dt + \sigma\, dW_t \right]$$

$\Rightarrow$ $A = a + \frac{\sigma^2}{2}$

$B = \sigma$

#21. ②. Find sufficient and necessary conditions on a and σ for S_t to be mtg.

Pick $s<t$ arbitrarily, then we have:

$E(S_t \mid \mathcal{F}_s) = E(x_0 e^{at} e^{\sigma W_t} \mid \mathcal{F}_s)$ where $(W_t, \mathcal{F}_t)$ is Wiener.

$= x_0 e^{at} E(e^{\sigma W_t} \mid \mathcal{F}_s)$ as $x_0 e^{at}$ are non-random.

$= x_0 e^{at} e^{\sigma W_s} E(e^{\sigma(W_t - W_s)} \mid \mathcal{F}_s)$ as $e^{\sigma W_s}$ is $\mathcal{F}_s$-measurable

$$E(e^{\sigma(W_t-W_s)} \mid \mathcal{F}_s) = \int_{-\infty}^{+\infty} e^{\sigma z} \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-z^2}{2(t-s)}} dz \quad \text{as} \begin{cases} W_t \text{ is Markov} \\ W_t - W_s \sim N(0, t-s) \end{cases}$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z^2}{(t-s)} - 2\sigma z\right)} dz$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z}{\sqrt{t-s}} - \sqrt{t-s}\,\sigma\right)^2 + \frac{1}{2}(t-s)\sigma^2} dz$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{1}{2}(t-s)\sigma^2} \int_{-\infty}^{+\infty} e^{-\frac{(z-(t-s)\sigma)^2}{2(t-s)}} dz$$

But $d(z-(t-s)\sigma) = dz \Rightarrow \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{+\infty} e^{-\frac{(z-(t-s)\sigma)^2}{2(t-s)}} dz = 1$

$\Rightarrow E(e^{\sigma(W_t-W_s)} \mid \mathcal{F}_s) = e^{\frac{1}{2}(t-s)\sigma^2}$

$E(S_t \mid \mathcal{F}_s) = x_0 e^{at} e^{\sigma W_s} e^{\frac{1}{2}(t-s)\sigma^2}$

For $(S_t, \mathcal{F}_t)$ to be a mtg. $\Leftrightarrow$ $E(S_t \mid \mathcal{F}_s) = S_s$, if $s<t$

$E(S_t \mid \mathcal{F}_s) = x_0 e^{at} e^{\sigma W_s} e^{\frac{1}{2}(t-s)\sigma^2} = S_s e^{a(t-s)} e^{\frac{1}{2}(t-s)\sigma^2} = S_s$

iff $a = -\frac{1}{2}\sigma^2$

Therefore, $a = -\frac{1}{2}\sigma^2$ is the sufficient & necessary condition for $(S_t, \mathcal{F}_s)$ to be a mtg.

#22. Given $(w_t, \mathcal{F}_t)$ a Wiener process, $(X_t, \mathcal{F}_t)$ a bounded process,

WTS: $Z_t := \exp\left(\int_0^t X_s dW_s - \frac{1}{2}\int_0^t X_s^2 ds\right)$ is a local mtg. w.r.t. $(\mathcal{F}_t)$

Proof: Let $f: \mathbb{R} \to \mathbb{R}$ be $f(\xi) = e^{\xi}$, and

$$\xi_t := \int_0^t X_s dW_s - \frac{1}{2}\int_0^t X_s^2 ds \iff d\xi_t = X_t dW_t - \frac{1}{2} X_t^2 dt$$

By Itô's Rule, since ξ_t is a semi-martingale w/ $\begin{cases} \int_0^t X_s dW_s \in \mathcal{M}^{c,loc} \\ \int_0^t X_s^2 ds \text{ is continuous and has bdd variation.} \end{cases}$

we have: $Z_t = f(\xi_t) = \underset{①}{f(\xi_0)} + \underset{②}{\int_0^t f'(\xi_s) d\xi_s} + \underset{③}{\frac{1}{2}\int_0^t f''(\xi_s) d\langle w\rangle_s}$

①. $f(\xi_0) = \exp(0) = 1$

②. $\int_0^t f'(\xi_s) d\xi_s = \int_0^t Z_s (X_s dW_s - \frac{1}{2} X_t^2 dt)$

③. $\frac{1}{2}\int_0^t f''(\xi_s) d\langle\xi\rangle_s = \frac{1}{2}\int_0^t Z_s X_s^2 ds$

$\Rightarrow Z_t = 1 + \int_0^t Z_s X_s dW_s$

It remains to show $\int_0^t Z_s X_s dW_s \in \mathcal{M}^{c,loc}$

Since $Z_s X_s = X_s \exp\left(\int_0^s X_r dW_r - \frac{1}{2}\int_0^s X_r^2 dr\right)$, we have:

$\forall \omega \in \Omega$, $\int_0^t |Z_s X_s|^2 ds \le K^2 \int_0^t e^{2K W_s(\omega)} e^{-K^2 s} ds \quad \Leftarrow |X| \le K, \forall \omega, \forall t$

$\le K^2 \max_{s\in[0,t]} e^{-K^2 s} \max_{s\in[0,t]} e^{2K W_s(\omega)} \int_0^t ds$ by continuity of $s \to W_s(\omega)$.

$< \infty$

$\Rightarrow P\left(\int_0^t |Z_s X_s|^2 ds < \infty\right) = 1 \Rightarrow (Z_t X_t, \mathcal{F}_t) \in \mathcal{P}(w)$ (*)

$I_t^W(ZX) = \int_0^t Z_s X_s dW_s \in \mathcal{M}^{c,loc} \quad \Leftarrow$ (*) & $(W_t) \in \mathcal{M}^{c,loc}$

$\square$