HW #2 Shizhou Xu Limit Theorems II

#1. WTS: $\forall h \in H$ is Gaussian, where H is the Hilbert space generated by the Gaussian process $(X_t)_{t\in\mathbb{R}}$.

Proof: Let $\hat{H} := \{\sum_{i=1}^{n} c_i X_{t_i} : c_i \in \mathbb{C}, t_i \in \mathbb{R}, n \in \mathbb{N}, n < \infty\}$

Pick $\hat{h}$ arbitrary from $\hat{H}$, then $\hat{h} = \sum_{i=1}^{n} c_i X_{t_i}$.

Since $(X_t)_{t\in\mathbb{R}}$ is a Gaussian process, $(X_{t_1}, \cdots, X_{t_n})^T$ is a Gaussian vector, and therefore $\hat{h} = \sum_{i=1}^{n} c_i X_{t_i} = \langle (c_1, \cdots, c_n)^T, (X_{t_1}, \cdots, X_{t_n})^T \rangle$ is Gaussian. (by #13 in Assignment 1).

Now, for $\forall h \in H$, $\exists (h_i)_{i\in\mathbb{N}} \subset \hat{H}$ s.t. $h_i \xrightarrow{L^2} h$

Since $\forall i \in \mathbb{N}$ $h_i \in \hat{H}$ is Gaussian, let $E|h_i|^2 =: \sigma_i^2$.

$$E e^{i\xi h_i} = \exp\left(-\tfrac{1}{2}\sigma_i^2 \xi^2\right).$$

Now, $h_i \xrightarrow{L^2} h \Rightarrow E|h_i|^2 \xrightarrow{i\to\infty} E|h|^2 =: \sigma^2$

$\Rightarrow E^{i\xi h_i} = \exp(-\frac{1}{2}\sigma_i^2\xi^2) \xrightarrow[i\to\infty]{} \exp(-\frac{1}{2}\sigma^2\xi^2)$ pointwise

By Levy's Theorem, $\exists \hat{h} \in H$ s.t. $E^{i\xi\hat{h}} = \exp(-\frac{1}{2}\sigma^2\xi^2)$

But since L^2 convergence implies convergence in distribution,

By the uniqueness of limit, we have $\hat{h} = h$ a.s.

$\Rightarrow E^{i\xi h} = \exp(-\frac{1}{2}\sigma^2\xi^2) \Rightarrow h \sim \mathcal{N}(0, \sigma^2)$

$\square$

#2. Given ϕ independent of (A, η) and $\sim U[0, 2\pi]$,

WTS: $(X_t)_{t \in \mathbb{R}}$ w/ $X_t = A\cos(\eta t + \phi)$, $t \in \mathbb{R}$ is strictly stationary.

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Proof:

①. Pick an arbitrary $t \in \mathbb{R}$, we have $X_t = A\cos(\eta t + \phi)$.

Since cos is a continuous function, A, η, ϕ are R.V.'s,

we have $\cos(\eta t + \phi)$ is a R.V. and $A\cos(\eta t + \phi)$ is a R.V.

$\Rightarrow (X_t)_{t \in \mathbb{R}}$ is a stochastic process.

②. Pick $h \in \mathbb{R}$ be arbitrary, choose event $\{\eta = \bar{\eta}\}$, we have:

$X(t+h) = A\cos[\bar{\eta}(t+h) + \phi] = A\cos(\bar{\eta} t + \bar{\phi}_h)$

where $\bar{\phi}_h := (\bar{\eta} h + \phi) \mod 2\pi$

Since the map $\phi \to \bar{\phi}_h$ is one-to-one and $\phi \sim U[0, 2\pi]$,

$\bar{\phi}_h \sim U[0, 2\pi]$.

Since $\phi \perp (A, \eta)$, $\bar{\phi}_h \sim U[0, 2\pi]$ for $\{(A, \eta) \in (B_1, B_2)\}$, $\forall B_1, B_2 \in \mathcal{B}(\mathbb{R})$

$\Rightarrow P\{\bar{\eta}(t+h) + \phi \in B\} = P\{\eta t + \bar{\phi}_h \in B\}$ $\forall B \in \mathcal{B}(\mathbb{R})$.

$\Rightarrow A\cos(\eta(t+h) + \phi) \stackrel{d}{=} A\cos(\eta t + \bar{\phi}_h)$, $\forall t \in \mathbb{R}$.

Now, pick $\{t_i\}_{i=1}^{n} \subset \mathbb{R}$, and $\{B_i\}_{i=1}^{n} \in \mathcal{B}(\mathbb{R})$ arbitrarily,

we have:

$$P\{X_{t_i+h}\in B_i,\ i\in\{1,\cdots,n\}\}$$

$$=P\{A\cos(\eta(t_i+h)+\phi)\in B_i,\ i\in\{1,\cdots,n\}\}$$

$$=P\{A\cos(\eta t_i+\phi_h)\in B_i,\ i\in\{1,\cdots,n\}\}$$

$$=P\{A\cos(\eta t_i+\phi)\in B_i,\ i\in\{1,\cdots,n\}\}$$

$$=P\{X_{t_i}\in B_i,\ i\in\{1,\cdots,n\}\}$$

① & ② $\Rightarrow$ $(X_t)_{t\in\mathbb{R}}$ is a strictly stationary process.

$\square$

#4. WTS: $\exists$ a covariance function $C(t)$ s.t. the spectral measure

#4. $S(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} C(t)\,dt$ equals to $\frac{1}{1+x^2}$

Let $C(t) := a e^{-b|t|}$, $a, b \in \mathbb{R}^+$.

$$S(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} a e^{-b|t|}\,dt$$

$$= \frac{a}{2\pi}\Big(\underbrace{\int_{-\infty}^{0} e^{(-ix+b)t}\,dt}_{①} + \underbrace{\int_{0}^{\infty} e^{(-ix-b)t}\,dt}_{②}\Big)$$

① $\int_{-\infty}^{0} e^{(-ix+b)t}\,dt = \frac{1}{-ix+b} e^{(-ix+b)t}\Big|_{-\infty}^{0} = \frac{1}{-ix+b}\left(1 - \lim_{t\to-\infty} e^{(-ix+b)t}\right)$

where $\lim_{t\to-\infty} e^{(-ix+b)t} = \lim_{t\to-\infty} e^{bt}\left(\cos(xt) - i\sin(xt)\right) = 0$.

② $\int_{0}^{\infty} e^{(-ix-b)t}\,dt \overset{\tilde{t}=-t}{=} \int_{0}^{-\infty} e^{(ix+b)\tilde{t}}(-1)\,d\tilde{t} = \int_{-\infty}^{0} e^{(ix+b)\tilde{t}}\,d\tilde{t}$

$$= \frac{1}{ix+b} e^{(ix+b)}\Big|_{-\infty}^{0} = \frac{1}{ix+b}$$

$$S(x) = \frac{a}{2\pi}\left(\frac{1}{-ix+b} + \frac{1}{ix+b}\right) = \frac{a}{2\pi}\cdot\frac{2}{x^2+b^2} = \frac{a}{\pi(x^2+b^2)}$$

Let $a=\pi$, $b=1$, we have $C(t) = \pi e^{-|t|}$, $S(x) = \frac{1}{1+x^2}$

Since $C(t) = \pi e^{-|t|} = \langle \sqrt{\pi}$

it is a covariance function and satisfying $C(t) = \int_{\mathbb{R}} e^{itx}\frac{dx}{1+x^2}$

$\square$

#5. WTS: $\exists$ non-ergodic weakly stationary process.

#[redacted]5.

Example: Let $\Omega = \{0,1\}$, $\mathcal{F} = \{\phi, \{0\}, \{1\}, \{0,1\}\}$,

P: $P(\{0\}) = P(\{1\}) = \frac{1}{2}$, $P(\phi) = 0$, $P(\Omega) = 1$

Let $\begin{cases} X_0 : X_0(\{i\}) = i, \ i \in \{0,1\} \\ \theta(A) = A, \ \forall A \in \mathcal{F} \end{cases}$ $X_i := X_0(\theta^n(A))$, $\forall A \in \mathcal{F}$

①. Verify θ is P-preserving:

$$P(\theta^{-1}(A)) = P(A) = \begin{cases} \frac{1}{2} & \text{if } A = \{X_{\theta i} \in \{0\}, \forall i\} \text{ or } A = \{X_{\theta i} \in \{1\}, \forall i\} \\ 0 & \text{otherwise} \end{cases}$$

②. Verify $(X_n)_{n \in \mathbb{N} \cup \{0\}}$ is weakly stationary:

1. $E(X_n) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$ for $\forall n \in \mathbb{N} \cup \{0\}$.
2. $E(X_i - \frac{1}{2})(X_j - \frac{1}{2}) = \frac{1}{2}(1-\frac{1}{2})(1-\frac{1}{2}) + \frac{1}{2}(0-\frac{1}{2})(0-\frac{1}{2}) = \frac{1}{4}$
which is a constant function $C(i-j) = \frac{1}{4}$

By 1 & 2, $(X_n)_{n \in \mathbb{N} \cup \{0\}}$ is weakly stationary.

③. Show $(X_n)_{n \in \mathbb{N} \cup \{0\}}$ is not ergodic

By ergodic theory we have $\sum_{i=0}^{n-1} \frac{1}{n} X_i(\omega) \xrightarrow[n \to \infty]{a.s.} E(X_0 | \mathcal{I})$

$\mathcal{I} := \{\phi, \{0\}, \{1\}, \{0,1\}\} = \mathcal{F}$

This sort of works but the question was about convergence in L^2

$\Rightarrow E(X_0 | \mathcal{F}) = X_0$ is a non-constant R.V.

$\Rightarrow (X_n)_{n \in \mathbb{N} \cup \{0\}}$ is not ergodic

$\square$

#6. WTS: $Y_n := \sum_{k=-K}^{K} a_k X_{n+k}$ is weakly stationary, given $(X_n)_{n\in\mathbb{Z}}$ is weakly stationary, for $\forall$ $K \in \mathbb{N}$, $\forall$ $(a_i)_{i\in\{-K,\dots,K\}} \subset \mathbb{C}$.

#6.

Proof: Pick arbitrary $K \in \mathbb{N}$, arbitrary $(a_i)_{i\in\{-K\cdots K\}} \subset \mathbb{C}$

Since $(X_n)_{n\in\mathbb{Z}}$ is weakly stationary, we have

$E X_n = m$. $\forall$ $n \in \mathbb{Z}$.

$$E(X_n - m)\overline{(X_m - m)} = E X_n \overline{X_m} - 2|m|^2 + |m|^2 = E X_n \overline{X_m} - |m|^2$$
$$= \gamma(n-m)$$

$$\Rightarrow E X_n \overline{X_m} = \gamma(n-m) + |m|^2$$

①
$$E(Y_n) = \sum_{k=-K}^{K} a_k E X_{n+k} = m\left(\sum_{k=-K}^{K} a_k\right)$$

②.
$$E(Y_n - EY_n)\overline{(Y_m - EY_m)} = E Y_n \overline{Y_m} - \left|m\left(\sum_{k=-K}^{K} a_k\right)\right|^2$$, where

$$E Y_n \overline{Y_m} = E\left(\sum_{k=-K}^{K} a_k X_{n+k}\right)\overline{\left(\sum_{j=-K}^{K} a_j X_{m+j}\right)} = \sum_{k=-K}^{K}\sum_{j=-K}^{K} a_k \overline{a_j}\left[\gamma(n-m+k-j) + |m|^2\right]$$

$$= \left(\gamma(n-m) + |m|^2\right)\left(\sum_{k=-K}^{K} a_k \overline{a_k}\right) + \left(\gamma(n-m+1) + |m|^2\right)\left(\sum_{k=-K+1}^{K} a_k \overline{a_{k-1}}\right)$$
$$+ \cdots + \left(\gamma(n-m+2K-1) + |m|^2\right)\left(\sum_{k=-K+(2K-1)}^{K} a_k \overline{a_{k-(2K-1)}}\right)$$
$$+ \left(\gamma(n-m+2K) + |m|^2\right) a_K \overline{a_{-K}} + \left(\gamma(n-m-1) + |m|^2\right)\left(\sum_{k=-K+1}^{K} a_{k-1} \overline{a_k}\right)$$
$$+ \cdots + \left(\gamma(n-m-(2K-1)) + |m|^2\right)\left(\sum_{k=-K+2K-1}^{K} a_{k-(2K-1)} \overline{a_k}\right) + \left(\gamma(n-m-2K) + |m|^2\right)|a_K|^2$$
$$= \sum_{i=0}^{K} \gamma(n-m+i)\left(\sum_{k=-K+i}^{K} a_k \overline{a_{k-i}}\right) + \sum_{i=-1}^{-K} \gamma(n-m+i)\left(\sum_{k=-K-i}^{K} a_{k+i} \overline{a_k}\right)$$
$$+ |m|^2 \left|\sum_{k=-K}^{K} a_k\right|^2$$

$$E\,(Y_n - EY_n)\overline{(Y_m - EY_m)} = \underbrace{\sum_{i=0}^{K} f(n-m+i)\Big(\sum_{k=-K+i}^{K} a_k \bar{a}_{k-i}\Big) + \sum_{i=-1}^{-K} f(n-m+i)\Big(\sum_{k=-K-i}^{K} a_{k+i}\bar{a}_k\Big)}_{\overset{!!\text{ define}}{C_{(a_k)_{-K}^{K}}(n-m)}}$$

where $C_{(a_k)_{-K}^{K}}(n-m)$ is a function of $(n-m)$ given fixed $(a_k)_{k=-K}^{K}$.

By ① & ②, $(Y_n)_{n\in\mathbb{Z}}$ is weakly stationary.

Since our choices of $K \in \mathbb{N}$, $(a_k)_{k=-K}^{K} \subset \mathbb{C}$ are arbitrary, we're done

$\square$

WTS: $\rho_Y(dx) := f[\rho_X(dx)]$

$$C_Y(t) = \sum_{j=0}^{K}\Big(\sum_{k=-K+j}^{K} a_k\bar{a}_{k-j}\Big) C_X(t+j) + \sum_{j=-1}^{-K}\Big(\sum_{k=-K-j}^{K} a_{k+j}\bar{a}_k\Big) C_X(t+j)$$

$$C_X(t) = \int e^{it\lambda}\rho_X(d\lambda) \Rightarrow C_X(t+j) = \int e^{it\lambda}e^{ij\lambda}\rho_X(d\lambda).$$

$$C_Y(t) = \int e^{it\lambda}\Big[\sum_{j=0}^{K}\Big(\sum_{k=-K+j}^{K} a_k\bar{a}_{k-j}\Big)e^{ij\lambda} + \sum_{j=-1}^{-K}\Big(\sum_{k=-K-j}^{K} a_{k+j}\bar{a}_k\Big)e^{ij\lambda}\Big]\rho_X(d\lambda)$$

$$\Rightarrow \rho_Y(d\lambda) = \Big[\sum_{j=0}^{K}\Big(\sum_{k=-K+j}^{K} a_{k+j}\bar{a}_k\Big)e^{ij\lambda} + \sum_{j=-1}^{-K}\Big(\sum_{k=-K-j}^{K} a_{k+j}\bar{a}_k\Big)e^{ij\lambda}\Big]\rho_X(d\lambda).$$

#7. Describe $(X_n)_{n\in\mathbb{Z}}$ w/ $X_n := \int_0^{2\pi} \cos(n\lambda)\, Z(d\lambda)$, $n \in \mathbb{Z}$, where $Z(\cdot)$ is the standard white noise on $[0, 2\pi)$.

$$C(n) := E(X_n \bar{X}_0)$$

$$= E\left(\int_0^{2\pi} \cos(n\lambda)\, Z(d\lambda) \int_0^{2\pi} 1\, \overline{Z(d\hat{\lambda})} \right)$$

$$= \int_0^{2\pi}\int_0^{2\pi} \cos(n\lambda)\, E\left(Z(d\lambda)\overline{Z(d\hat{\lambda})}\right)$$

$$= \int_0^{2\pi}\int_0^{2\pi} \cos(n\lambda)\, \delta(\lambda-\hat{\lambda})\, d\lambda\, d\hat{\lambda}$$

$$= \int_0^{2\pi} \cos(n\lambda)\, d\lambda$$

$$= \frac{1}{n} \sin(n\lambda)\Big|_0^{2\pi} = 0, \quad \forall n \neq 0$$

$\Rightarrow$ X_n's are uncorrelated.

Sorry for my lack of knowledge of the White Noise.
Is there a good book from where I could study the White Noise and it as the "derivative" of Wiener process?

I am not sure what exactly seems to be a problem. Perhaps we should talk.

#8.

WTS: $P\{\lim_{n\to\infty} \frac{X_n}{n} = 0\} = 1$, given $(X_n)_{n\in\mathbb{Z}}$ is stationary w/ $EX_0 < \infty$

Proof: Since $(X_n)_{n\in\mathbb{Z}}$ is stationary, we can define

$$\theta: \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}} \text{ by } \theta X_\cdot(\omega) = X_{\cdot+1}(\omega), \ \forall\, \omega\in\Omega.$$

$$\xi: \mathbb{R}^{\mathbb{Z}} \to \mathbb{R} \text{ by } \xi X(\omega) = X_0(\omega), \ \forall\, \omega\in\Omega.$$

$\Rightarrow$ θ is measure preserving on $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), PX^{-1})$

Since $\forall\, n\in\mathbb{N}$, $X_n(\omega) = (\theta^n X_0)(\omega)$, $\forall\, \omega\in\Omega$, and $E(\xi X) = E(X_0) < \infty$

$$\sum_{i=0}^{n-1} \frac{X_i}{n} = \sum_{i=0}^{n-1} \frac{\theta^i X_0}{n} \xrightarrow{a.s.} E(\xi X \mid \mathcal{I}) \text{ by Ergodic Theorem}$$

where $\mathcal{I} := \{X_\cdot(\omega)\in\mathbb{R}^{\mathbb{Z}} : \theta^{-1} X_\cdot(\omega) = X_\cdot(\omega)\}$.

$$\Rightarrow \exists\, N \text{ s.t. } \begin{cases} \sum_{i=0}^{n-1} \frac{\theta^i X_0(\omega)}{n} \xrightarrow[n\to\infty]{} E(\xi X_\cdot(\omega)\mid\mathcal{I}), \ \forall\, X_\cdot(\omega)\in N \\ PX^{-1}(N) = 0 \end{cases}$$

Now, $\frac{X_n}{n} = \frac{X_0+\cdots+X_n}{n} - \frac{X_0+\cdots+X_{n-1}}{n}$

$$\Rightarrow \lim_{n\to\infty} \frac{X_n}{n} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{X_n}{n+1} \cdot \lim_{n\to\infty} \frac{n+1}{n} - \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{X_{n-1}}{n}$$

$$= E(\xi X\mid\mathcal{I}) - E(\xi X\mid\mathcal{I}) = 0 \text{ on } X^{-1}(N).$$

But $P(X^{-1}(N)) = PX^{-1}(N) = 0$

$\Rightarrow \lim_{n\to\infty} \frac{X_n}{n} = 0$ on $\Omega\setminus X^{-1}(N)$, $P(\Omega\setminus X^{-1}(N)) = 1$

$\Rightarrow P\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 1$

$\square$

#9. WTS: the spectral measure and spectral representation of

$X_t := A\cos(t+\phi)$, where $\phi \sim U[0,2\pi]$ independent of A.

①. Find covariance function for $(X_t)_{t\in\mathbb{R}}$:

$C(t) = E(X_t\bar{X}_0)$ by stationarity of $(X_t)_{t\in\mathbb{R}}$.

$= E(A\cos(t+\phi)\bar{A}\cos(\phi))$

$= E|A|^2\, E(\cos(t+\phi)\cos(\phi))$ by independence b/t A & $\cos(t+\phi)$

$= E|A|^2 \frac{1}{2} E(\cos(t+2\phi)+\cos(t))$

$= \frac{E|A|^2}{2}\left(\int_0^{2\pi}\cos(t+2\phi)\frac{1}{2\pi}d\phi + \int_0^{2\pi}\cos(t)\frac{1}{2\pi}d\phi\right)$ as $\phi\sim U[0,2\pi]$

$= \frac{E|A|^2}{4\pi}\left(\underbrace{\frac{1}{2}\sin(t+2\phi)\Big|_0^{2\pi}}_{=0} + 2\pi\cos t\right)$

$= \frac{E|A|^2}{2}\cos t$.

②. Find spectral measure:

$F'(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}C(t)\,dt = \frac{E|A|^2}{4\pi}\int_{\mathbb{R}} e^{-i\lambda t}\cos t\,dt$

$= \frac{E|A|^2}{4\pi}\int_{\mathbb{R}}\frac{1}{2}\left(e^{-i(\lambda+1)t}+e^{-i(\lambda-1)t}\right)dt$

$= \frac{E|A|^2}{4}\left(\delta(\lambda+1)+\delta(\lambda-1)\right)$

$\Rightarrow F(\lambda) = \frac{E|A|^2}{4}\left(H(\lambda+1)+H(\lambda-1)\right)$, H is the Heaviside function.

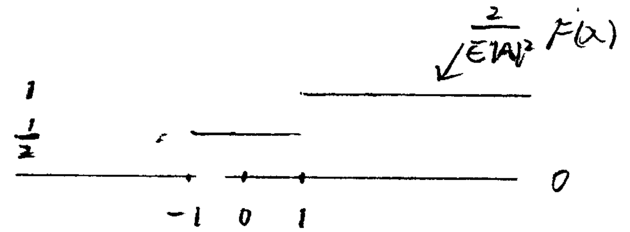
is the spectral measure.

③. Find the spectral representation

$$\frac{2}{E|A|^2} F(\lambda) = \frac{1}{2}\left(H(\lambda+1) + H(\lambda-1) \right)$$

$H(\lambda+1)$: 0 for $\lambda < -1$, 1 for $\lambda \geq 1$

$H(\lambda-1)$: 0 for $\lambda < 1$, 1 for $\lambda \geq 1$



Define $\quad S_\lambda := \frac{\|A\|_{L^2}}{2} \left(\delta(\lambda-1) + \delta(\lambda+1) \right) \mathbb{1}_{(-\infty, \lambda]}$ w/ $S_{-\infty} = 0$.

If $(\lambda_1, \lambda_2] \cap (\lambda_3, \lambda_4] = \phi$

$$E\left[(S_{\lambda_2} - S_{\lambda_1}) \overline{(S_{\lambda_4} - S_{\lambda_3})} \right] = E\left[\frac{E|A|^2}{4} \left(\delta(\lambda-1) + \delta(\lambda+1) \right) \mathbb{1}_{(\lambda_1, \lambda_2]} \mathbb{1}_{(\lambda_3, \lambda_4]} \right]$$

$$= 0$$

$$E\left(|S_{\lambda_2} - S_{\lambda_1}|^2 \right) = E\left[\frac{E|A|^2}{4} \left(\delta(\lambda-1) + \delta(\lambda+1) \right) \mathbb{1}_{(\lambda_1, \lambda_2]} \right]$$

$$= \frac{E|A|^2}{4} \left[\frac{1}{2} \mathbb{1}_{(\lambda_1, \lambda_2]}(-1) + \frac{1}{2} \mathbb{1}_{(\lambda_1, \lambda_2]}(1) \right]$$

$$= F(\lambda_2) - F(\lambda_1).$$

$\Rightarrow \quad X_t = \int_{\mathbb{R}} e^{i\lambda t} \, S(d\lambda)$ is the spectral representation.

#10. WTS: #10. $\{A \subset \Omega : \theta^{-1}A = A\}$ is a σ-algebra.

(i)

Proof: Let $\mathcal{A} := \{A \subset \Omega ; \theta^{-1}A = A\}$

① $\Omega \in \mathcal{A}$. If not, $\exists A \subset \Omega$, $A \neq \phi$ s.t. $\theta A = \phi$,

$A \neq \phi \Rightarrow \exists a \in A$, $\theta a \in \theta A \neq \phi \Rightarrow \rightarrow\leftarrow$ (contradiction).

②. Let $A \in \mathcal{A}$,

($\subset$) If $\hat{a} \in A^c$, $\hat{a} \notin (\theta^{-1}A) \Rightarrow \theta\hat{a} \in A^c$

$\Rightarrow \hat{a} \in \theta^{-1}A^c \Rightarrow A^c \subset \theta^{-1}A^c$

($\supset$) If $a \in A$, $a \in \theta^{-1}A \Rightarrow \theta a \in A \Rightarrow a \notin \theta^{-1}A^c$

$\Rightarrow A^c \supset \theta^{-1}A^c$

By ($\subset$) & ($\supset$), $A^c = \theta^{-1}A^c \Rightarrow A^c \in \mathcal{A}$.

③. Let $(A_i)_{i \in \mathbb{N}} \subset \mathcal{A}$, then $\theta^{-1}A_i = A_i$ for $\forall i \in \mathbb{N}$.

($\subset$) If $a \in \bigcup_{i=1}^{\infty} A_i$, $a \in A_i$ for some $i \in \mathbb{N}$, $a \in \theta^{-1}A_i$

$\Rightarrow \theta a \in A_i \subset \bigcup_{i=1}^{\infty} A_i \Rightarrow a \in \theta^{-1}\bigcup_{i=1}^{\infty} A_i$

Since $a \in \bigcup_{i=1}^{\infty} A_i$ is arbitrary, $\bigcup_{i=1}^{\infty} A_i \subset \theta^{-1}\bigcup_{i=1}^{\infty} A_i$

($\supset$) If $a \notin \bigcup_{i=1}^{\infty} A_i$, $a \notin A_i$ for any i, $a \notin \theta^{-1}A_i$ for any i

$\Rightarrow \theta a \notin A_i$ for any $i \Rightarrow a \notin \theta^{-1}\bigcup_{i=1}^{\infty} A_i$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \supset \theta^{-1}\bigcup_{i=1}^{\infty} A_i$

By ($\subset$) & ($\supset$), $\bigcup_{i=1}^{\infty} A_i = \theta^{-1}\bigcup_{i=1}^{\infty} A_i \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Finally, by ①, ②, ③, $\mathcal{A}$ is a σ-algebra. $\square$

(ii) Counterexample for $\{A \subset \Omega, \theta A = A\}$

Let $\Omega = \{1, 2\}$, $\theta(1) = 2$, $\theta(2) = 2$.

$\Rightarrow \Omega \notin \{A \subset \Omega, \theta A = A\}$

$\Rightarrow \{A \subset \Omega, \theta A = A\}$ is not an σ-algebra.

#11. A. WTS: $([0,1), \mathcal{B}([0,1)), m, \theta_\lambda)$ is ergodic, where θ_λ is defined by $\theta_\lambda w = \{w+\lambda\}$, iff $\lambda \notin \mathbb{Q}$

Proof: ($\Rightarrow$) Assume $\lambda \in \mathbb{Q}$, then we have that $\lambda = l + \frac{k}{n}$, $l, k, n \in \mathbb{Z}$, $k < n$.

Now, let $A := \bigcup_{i=0}^{n-1} [\frac{i}{n}, \frac{i}{n} + \frac{1}{nk})$, then

$m(\theta_\lambda [\frac{i}{n}, \frac{i}{n}+\frac{1}{nk})) = m([\{\frac{k+i}{n}\}, \{\frac{k+i}{n}\} + \frac{1}{nk})) = \frac{1}{nk}$, $\forall i = 0, \dots, n-1$

and $\theta_\lambda A = \sum_{i=0}^{n-1} [\{\frac{k+i}{n}\}, \{\frac{k+i}{n}\} + \frac{1}{nk}) = \sum_{i=0}^{n-1} [\frac{i}{n}, \frac{i}{n} + \frac{1}{nk}) = A$

$\Rightarrow \theta_\lambda^{-1} A = A$, $m(\theta_\lambda^{-1} A) = m(A) = \frac{1}{k}$

Since $A \notin \{\phi, [0,1)\}$, $([0,1), \mathcal{B}([0,1)), m, \theta_\lambda)$ is not ergodic

Now, if $k = n$, $\lambda \in \mathbb{Z}$, let $A := [\frac{1}{3}, \frac{2}{3})$, $\theta_\lambda^{-1} A = A$. $A \notin \{\phi, [0,1)\}$

$\Rightarrow \forall \alpha \in \mathbb{Q}$, $\exists A$ s.t. $\theta_\alpha^{-1} A = A$, $A \notin \{\phi, [0,1)\}$

$\Rightarrow \forall \alpha \in \mathbb{Q}$, $([0,1), \mathcal{B}([0,1)), m, \theta_\lambda)$ is not ergodic.

($\Leftarrow$) Assume $\alpha \notin \mathbb{Q}$, let $\mathbb{1}_B$ be a θ_λ-invariant function, $B \in \mathcal{B}([0,1))$

then $a_n = \int_B e^{2\pi i n x} dx = \int_B e^{2\pi i n (x+\alpha)} dx = e^{2\pi i n \alpha} a_n$

$\Rightarrow a_n = 0$, if $n \neq 0$ $\Rightarrow P(B|\mathcal{I})$ is constant a.s.

$\Rightarrow \mathcal{I} = \{A \in \mathcal{B}([0,1)) : P(A) = 0 \text{ or } 1\}$.

$\Rightarrow \forall \alpha \in \mathbb{Q}$, $([0,1), \mathcal{B}([0,1)), m, \theta_\lambda)$ is ergodic.

$\square$

#11. B. WTS: $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{f(\theta_\lambda^i(\omega))}{n}$

①. Let $f = \mathbb{1}_B$ for arbitrary $B \in \mathcal{B}([0,1))$, then by Birkhoff - Ergodic, we have:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\mathbb{1}_B(\theta_\lambda^i(\omega))}{n} = E(\mathbb{1}_B | \mathcal{I})$$

Now, by Part A, we also have:

$$E(\mathbb{1}_B|\mathcal{I}) = \begin{cases} \text{constant} & \text{if } \alpha \notin \mathbb{Q} \\ \text{non-constant} & \text{if } \lambda \in \mathbb{Q} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{f(\theta_\lambda^i(\omega))}{n} = E(f|\mathcal{I}) = \begin{cases} \text{constant} & \text{if } \alpha \notin \mathbb{Q} \\ \text{non-constant} & \text{if } \alpha \in \mathbb{Q}. \end{cases}$$

for $f = \mathbb{1}_B$, $B \in \mathcal{B}([0,1))$.

②. Let $f = \sum_{i=1}^{n} c_i \mathbb{1}_{B_i}$, by linearity of E and ①, we have:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{f(\theta_\lambda^i(\omega))}{n} = \sum_{i=1}^{n-1} c_i P(B_i|\mathcal{I}) = \begin{cases} \text{constant} & \text{if } \lambda \notin \mathbb{Q} \\ \text{non-constant} & \text{if } \lambda \in \mathbb{Q} \end{cases}$$

③. Let $f \in L^1$. Since $f \le |f| \in L^1$, by LDCT and ②

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{f(\theta_\lambda^i(\omega))}{n} = \lim_{n\to\infty} \sum_{i=0}^{n-1} c_i P(B_i|\mathcal{I}) = \begin{cases} \text{constant} & \text{if } \lambda \notin \mathbb{Q} \\ \text{non-constant} & \text{if } \lambda \in \mathbb{Q}. \end{cases}$$

$$= E(f|\mathcal{I})$$

$\square$

\# 12 : WTS : Every Gaussian martingale is a process w/ independent increments.

Proof: Let $((M_t)_{t\in\pi}, (\mathcal{F}^M_t)_{t\in\pi})$ be an arbitrary Gaussian martingale.

Since $\forall\ t \in \pi$, M_t is adapted to $\mathcal{F}^M_t \subset \bigcup_{t\in\pi} \mathcal{F}^M_t =: \mathcal{F}^M$

we have M_t is a $\mathcal{F}^M$-measurable R.V.

$\Rightarrow$ $(M_t)_{t\in\pi}$ is a process.

Now, for arbitrary $0 \le s < t$, $s, t \in \pi$, we have

$$E[(M_t - M_s) M_s] = E\left(E[(M_t - M_s) M_s \mid \mathcal{F}^M_s]\right)$$

$$= E\left(M_s\, E[(M_t - M_s) \mid \mathcal{F}^M_s]\right) \text{ as } M_s \text{ is } \mathcal{F}^M_s\text{-measurable}$$

But $E(M_t - M_s \mid \mathcal{F}^M_s) = E(M_t \mid \mathcal{F}^M_s) - M_s$ as M_s is $\mathcal{F}^M_s$-measurable

$= M_s - M_s$ by martingale property

$= 0$

$\Rightarrow$ $E[(M_t - M_s) M_s] = 0$, $\forall\ 0 \le s < t$, $s, t \in \pi$

$\Rightarrow$ $M_t - M_s$ and M_s are uncorrelated, $\forall\ 0 \le s < t$, $s, t \in \pi$

$\Rightarrow$ $M_s - M_s \perp M_s$ by Gaussianity (uncorrelation = independence).

$\forall$, $0 \le s < t$, $s, t \in \pi$

$\square$

#13 : WTS: $W_t^2 - t$ is a martingale w.r.t. $(\mathcal{F}_t^W)$

#13.

Proof: ①. By the definition of $\mathcal{F}_t^W := \sigma(X_s, s \le t)$, we have:

$$\forall t_1, t_2 \ge 0,\ t_1 \le t_2 \Rightarrow \sigma(X_s, s \le t_1) \subset \sigma(X_s, s \le t_2)$$
$$\Rightarrow \mathcal{F}_{t_1}^W \subset \mathcal{F}_{t_2}^W \Rightarrow (\mathcal{F}_t^W)_{t \ge 0} \text{ is a filtration.}$$

②. Since $(W_t)_{t\ge0}$ is a martingale w.r.t. $(\mathcal{F}_t^W)_{t\ge0}$, we have W_t is $\mathcal{F}_t^W$-measurable for $\forall t \ge 0$ (trivial actually)

Since $x \to x^2$ and $t \to t$ are both measurable as they are continuous on $\mathbb{R}$, we have:

$$(W_t)^2 - t \text{ is } \mathcal{F}_t^W\text{-measurable}, \ \forall t \ge 0$$

③. $E|W_t^2 - t| = E W_t^2 - E t$

? $= (t \wedge t) - t = 0 < \infty$

$$E(W_t^2 - t \mid \mathcal{F}_s^W) = E(W_t^2 \mid \mathcal{F}_s^W) - E(t \mid \mathcal{F}_s^W)$$
$$= E[(W_t^2 - W_s^2) + W_s^2 \mid \mathcal{F}_s^W] - E[(t-s)+s \mid \mathcal{F}_s^W]$$
$$= \underbrace{E(W_t^2 - W_s^2 \mid \mathcal{F}_s^W)}_{(i)} - \underbrace{E(t-s \mid \mathcal{F}_s^W)}_{(ii)} + (W_s^2 - s).$$

(i): $E(W_t^2 - W_s^2 \mid \mathcal{F}_s^W) = E[(W_t - W_s)^2 + 2W_tW_s - 2W_s^2 \mid \mathcal{F}_s^W]$

$$= E[(W_t - W_s)^2 \mid \mathcal{F}_s^W] + 2E[(W_t - W_s)W_s \mid \mathcal{F}_s^W]$$
$$= (t-s) + 2W_s E(W_t - W_s)$$
$$= t - s$$

(ii): $E(t-s \mid \mathcal{F}_s^W) = t - s.$

$\Rightarrow E(W_t^2 - t \mid \mathcal{F}_s^W) = W_s^2 - s$.

Now, by ①, ②, ③, we have $W_t^2 - t$ is a martingale w.r.t. $(\mathcal{F}_t^W)$

$\square$

#14. WTS: $P(s,x,t,\Gamma) = P(t-s, x, \Gamma) = \int_\Gamma \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} dy$

is a Markov transition probability function for the standard W.P.

Proof: ①. Note: $\int_\Gamma \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} dy = P(W_{t-s} + x \in \Gamma)$

$= P[(W_t - W_s) + W_s \in \Gamma \mid W_s = x]$

$= P(W_t \in \Gamma \mid W_s = x)$.

For $\forall s, t, \Gamma$, we have $P(s, W_s, t, \Gamma) \overset{a.s.}{=} P(W_t \in \Gamma \mid W_s)$

②. For $\forall s, t, \Gamma$ (except $s=t$, which we talk in ④),

$P(s,x,t,\Gamma) = \int_\Gamma \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} dy$ is continuous $\Rightarrow \mathcal{B}$-measurable.

③. For fixed s, x, t, we have:

$P(s,x,t,\Gamma) = \int_\Gamma \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} dy$

$= P_X(\Gamma)$, $X \sim N(x, t-s)$

Since $P_X: \mathbb{R} \to [0,1]$ is a probability measure

$P(s,x,t,\cdot) = P_X(\cdot)$ is a probability measure.

④. When $s=t$, we have

$P(s,x,s,\Gamma) = P_X(\Gamma)$, where $X \equiv x$.

$= \delta_x$, which is also $\mathcal{B}$-measurable.

Now, by ①, ②, ③, ④, $\int_\Gamma \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} dy$ is a Markov transition probability function for the standard W.P. $(W_t)_{t\in T}$

□

#15. WTS: $\forall A \in \mathcal{F}_{\le t}, B \in \mathcal{F}_{\ge t}$, $P(AB|X_t) = P(A|X_t)P(B|X_t)$

$\Leftrightarrow \forall B \in \mathcal{F}_{\ge t}$, $P(B|\mathcal{F}_{\le t}) = P(B|X_t)$.

Proof:

($\Leftarrow$) Assume $\forall B \in \mathcal{F}_{\ge t}$, $P(B|\mathcal{F}_{\le t}) = P(B|X_t)$ (*).

$$
\begin{aligned}
P(AB|X_t) &= E(\mathbb{1}_A \mathbb{1}_B | X_t) \\
&= E[E(\mathbb{1}_A\mathbb{1}_B | \mathcal{F}_{\le t}) | X_t] \quad \text{by tower property.} \\
&= E[\mathbb{1}_A E(\mathbb{1}_B | \mathcal{F}_{\le t}) | X_t] \quad \text{as } \mathbb{1}_A \text{ is } \mathcal{F}_{\le t}\text{-measurable.} \\
&= E[\mathbb{1}_A E(\mathbb{1}_B | X_t) | X_t] \quad \text{by (*)}. \\
&= E(\mathbb{1}_B | X_t)\, E(\mathbb{1}_A | X_t) \quad \text{as } E(\mathbb{1}_B|X_t) \text{ is } \sigma(X_t)\text{-measurable.} \\
&= P(B|X_t)\, P(A|X_t).
\end{aligned}
$$

($\Rightarrow$) Now, assume that $\forall A \in \mathcal{F}_{\le t}, B \in \mathcal{F}_{\ge t}$, we have
$P(AB|X_t) = P(A|X_t)P(B|X_t)$ (**).

$$
\begin{aligned}
E[\mathbb{1}_A P(B|X_t) | X_t] &= P(A|X_t)P(B|X_t) \quad \text{as } P(B|X_t) \text{ is } \sigma(X_t) \text{ measurable} \\
&= P(AB|X_t) \quad \text{by (**)}. \\
&= E(\mathbb{1}_A\mathbb{1}_B | X_t)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow E[\mathbb{1}_A P(B|X_t)] &= E[E(\mathbb{1}_A P(B|X_t)|X_t)] \\
&= E[E(\mathbb{1}_A\mathbb{1}_B | X_t)] = E(\mathbb{1}_A \mathbb{1}_B)
\end{aligned}
$$

Since this is true for $\forall A \in \mathcal{F}_{\le t}$, we have

$$
\begin{aligned}
E[P(B|X_t) | \mathcal{F}_{\le t}] &= E(\mathbb{1}_B | \mathcal{F}_{\le t}) \\
P(B|X_t) &= P(B|\mathcal{F}_{\le t}) \quad \text{as } \sigma(X_t) \subset \mathcal{F}_{\le t}
\end{aligned}
$$

$\square$

#16. WTS: $(X_t)_{t\le 0}$ is a Markov process, given $W_{-t} =: X_t$. Find the T.P.F for $(X_t)_{t\le 0}$

Proof: ① Claim: $A\in \mathcal{F}_{\le t}$, $B\in\mathcal{F}_{\ge t}$, then $P(B|\mathcal{F}_{\le t}) = P(B|X_t) \Leftrightarrow P(A|\mathcal{F}_{\ge t}) = P(A|X_t)$.

Proof: If $P(B|\mathcal{F}_{\le t}) = P(B|X_t)$, then $P(AB|X_t) = P(A|X_t)P(B|X_t)$ by #15

$$E(\mathbb{1}_B E(\mathbb{1}_A|X_t)|X_t) = E(\mathbb{1}_B|X_t)E(\mathbb{1}_A|X_t) = E(\mathbb{1}_A\mathbb{1}_B|X_t)$$

$$\Rightarrow E[\mathbb{1}_B E(\mathbb{1}_A|X_t)] = E(E[\mathbb{1}_B E(\mathbb{1}_A|X_t)|X_t]) = E[E(\mathbb{1}_A\mathbb{1}_B|X_t)] = E(\mathbb{1}_A\mathbb{1}_B)$$

$$\Rightarrow E(\mathbb{1}_A|X_t) = E[E(\mathbb{1}_A|X_t)|\mathcal{F}_{\ge t}] = E(\mathbb{1}_A|\mathcal{F}_{\ge t}).$$

Now, if $P(A|X_t) = P(A|\mathcal{F}_{\ge t})$, then:

$$E(\mathbb{1}_A\mathbb{1}_B|X_t) = E[E(\mathbb{1}_A\mathbb{1}_B|\mathcal{F}_{\ge t})|X_t] = E[\mathbb{1}_B E(\mathbb{1}_A|\mathcal{F}_{\ge t})|X_t]$$
$$= E(\mathbb{1}_B E(\mathbb{1}_A|X_t)|X_t) = E(\mathbb{1}_B|X_t)E(\mathbb{1}_B|X_t).$$

$\Rightarrow P(B|X_t) = P(B|\mathcal{F}_{\le t})$ by #15

Hence, $P(A|X_t) = P(A|\mathcal{F}_{\ge t}) \Leftrightarrow P(B|X_t) = P(B|\mathcal{F}_{\le t})$.

②. Pick $t\in(-\infty,0]$, since $\mathcal{F}^X_{\le t} = \mathcal{F}^W_{\ge -t}$, $\mathcal{F}^X_{\ge t} = \mathcal{F}^W_{\le -t}$, $\forall B\in\mathcal{F}^X_{\ge t}$.

$$E(\mathbb{1}_B|\mathcal{F}^X_{\le t}) = E(\mathbb{1}_B|\mathcal{F}^W_{\ge -t}) \underset{\text{by ①}}{=} E(\mathbb{1}_B|W_{-t}) = E(\mathbb{1}_B|X_t)$$

$\Rightarrow (X_t)_{t\le 0}$ is Markov.

□

Since $P_W(-s,x,-t,\Gamma) = \int_\Gamma \frac{1}{\sqrt{2\pi(s-t)}} e^{-\frac{(x-y)^2}{2(s-t)}}\,dy = P(W_{-t}\in\Gamma \mid W_{-s}=x)$,

we have $\int_\Gamma \frac{1}{\sqrt{2\pi(s-t)}} e^{-\frac{(x-y)^2}{2(s-t)}}\,dy =: P_X(s,x,t,\Gamma) = P(X_t\in\Gamma\mid X_s = x)$

Also, since $P_W(-s,\cdot,-t,\Gamma)$ is $\mathcal{B}$-measurable, so is $P_X(s,\cdot,t,\Gamma)$

$P_W(-s,x,-t,\cdot)$ is a P-measure, so is $P_X(s,x,t,\cdot)$

$P_W(-s,x,-s,\cdot) = \delta_x$, so is $P_X(s,x,s,\cdot)$

$\Rightarrow P_X(s,x,t,\Gamma)$, $t\le s$, $s,t\in(-\infty,0]$, $\Gamma\in\mathcal{B}(\mathbb{R})$, $x\in\mathbb{R}$ is a T.P.F. for $(X_t)_{t\le 0}$

#17. WTS: $(|W_t|)_{t \in \mathbb{T}}$ is Markov.

Proof: ①. $(|W_t|, \mathcal{F}_t^{|W|})$ satisfies $|W_t|$ is adapted to $\mathcal{F}_t^{|W|}$ trivially as $\mathcal{F}_t^{|W|} := \sigma(|W_s|, s \in [0,t])$.

②. For $\forall\, |W_s|^{-1}(B) \in \sigma(|W_s|)$, $|W_s|^{-1}(B) = (W_s \vee 0 - W_s \wedge 0)^{-1}(B) \in \sigma(W_s)$

$\Rightarrow \sigma(|W_s|) \subset \sigma(W_s)$.

$\Rightarrow \mathcal{F}_t^{|W|} := \sigma(|W_s|, s \in [0,t]) \subset \sigma(W_s ; s \in [0,t]) =: \mathcal{F}_t^{W}$.

Now, $E(|W_t| \in T \mid \mathcal{F}_s^{|W|})$, $0 \le s \le t$ be arbitrary in $\mathbb{T}$

$= E[E(|W_t| \in T \mid \mathcal{F}_s^{W}) \mid \mathcal{F}_s^{|W|}]$ Tower property

$= E[E(f(W_t) \in T \mid \mathcal{F}_s^{W}) \mid \mathcal{F}_s^{|W|}]$ $f = |\cdot|$ continuous

$= E[E(f(W_t) \in T \mid W_s) \mid \mathcal{F}_s^{|W|}]$ $(f(W_t), \mathcal{F}_t^{W})$ Markov.

$= E[E[E(f(W_t) \in T \mid |W_s|) \mid W_s] \mid \mathcal{F}_s^{|W|}]$

$= E[E[E(f(W_t) \in T \mid |W_s|) \mid \mathcal{F}_s^{W}] \mid \mathcal{F}_s^{|W|}]$ $(E(f(W_t) \in T \mid |W_s|), \mathcal{F}_s^{W})$ Markov

$= E(f(W_t) \in T \mid |W_s|)$ as $\sigma(|W_s|) \subset \mathcal{F}_s^{|W|} \subset \mathcal{F}_s^{W}$

$= E(|W_t| \in T \mid |W_s|)$

$\square$

#18. WTS: $(X_t)_{t\geq 0} := ((-1)^{N_t})_{t\geq 0}$ is a Markov process and find its T.P.F.

Proof: ①. Since $(N_t)_{t\geq 0}$ is a Poisson process ; N_t is $\sigma(\mathbb{N}^{\mathbb{T}})$ - measurable.

Since $\{(-1)^{N_t}\}_{t\geq 0} \subset \{-1, 1\}$, and $\begin{cases} [(-1)^{\cdot}]^{-1}(\{1\}) \in \sigma(\mathbb{N}^{\mathbb{T}}) \\ [(-1)^{\cdot}]^{-1}(\{-1\}) \in \sigma(\mathbb{N}^{\mathbb{T}}) \end{cases}$,

$(X_t)_{t\geq 0} = ((-1)^{N_t})_{t\geq 0}$ is $\sigma(\{-1,1\}^{\mathbb{T}})$ - measurable.

$\Rightarrow (X_t)_{t\geq 0}$ is a well-defined process.

②. Since

$$
\begin{aligned}
E(X_t \mid \mathcal{F}_s^N) &= E((-1)^{N_t} \mid \mathcal{F}_s^N) \\
&= E((-1)^{N_s}(-1)^{N_{t-s}} \mid \mathcal{F}_s^N) \\
\begin{cases} (-1)^{N_{t-s}} \perp \mathcal{F}_s^N \\ (-1)^{N_s} \text{ is } \mathcal{F}_s^N\text{-measurable} \end{cases} \Rightarrow &= (-1)^{N_s} E((-1)^{N_{t-s}}) \\
&= (-1)^{N_s} E((-1)^{N_{t-s}} \mid (-1)^{N_s}) \quad \text{as } (-1)^{N_{t-s}} \perp (-1)^{N_s} \\
&= E((-1)^{N_{t-s}}(-1)^{N_s} \mid (-1)^{N_s}) \\
&= E((-1)^{N_t} \mid (-1)^{N_s}) \\
&= E(X_t \mid X_s)
\end{aligned}
$$

By ①, ②, we have $(X_t)_{t\geq 0}$ is a Markov process.

Since $P(\{N_t = n\}) = \frac{t^n e^{-t}}{n!}$, and $\{X_{t-s} = 1\} = \{N_{t-s} \in \{2k, k\in\mathbb{N}\}\}$

$\{X_{t-s} = -1\} = \{N_{t-s} \in \{2k+1, k\in\mathbb{N}\}\}$

we have:

$$
\begin{aligned}
P(s,x,t,\Gamma) &= \sum_{k=0}^{\infty} \frac{(t-s)^{2k} e^{-(t-s)}}{2k!} \quad \text{if } \Gamma = \{x\} \\
&= \sum_{k=0}^{\infty} \frac{(t-s)^{2k+1} e^{-(t-s)}}{(2k+1)!} \quad \text{if } \Gamma = \{-x\}
\end{aligned}
$$

#19. WTS: density of τ_b and $E\tau_b$, given $\tau_b := \inf\{t \geq 0, W_t = b\}$

$$P\{\sup_{[0,T]} W_t > b\} = P\{\sup W_t > b, W_T \geq b\} + \sup\{\sup W_t \geq b, W_T < b\}.$$

$$= 2P\{\sup W_t > b, W_T > b\} \text{ by Markov and symmetry.}$$

$$= 2P\{W_T > b\}$$

$$P\{\tau_b < T\} = P\{\sup_{[0,T]} W_t > b\} = 2P\{W_T \geq b\}.$$

Since $P\{W_T \geq b\} = \int_b^{+\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx$

$$= \frac{1}{\sqrt{2\pi T}} \sqrt{2T} \int_{\frac{b}{\sqrt{2T}}}^{+\infty} e^{-\hat{x}^2} d\hat{x} \quad \text{by change of Var. } \hat{x} = \frac{x}{\sqrt{2T}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{b}{\sqrt{2T}}}^{+\infty} e^{-\hat{x}^2} d\hat{x}$$

By Libinz Rule, we have that

$$f_{\tau_b}(T) = \frac{d}{dT} 2P\{W_T > b\}$$

$$= \frac{2}{\sqrt{\pi}} \left[0 - \frac{b}{\sqrt{2}} \left(-\frac{1}{2}\right) T^{-\frac{3}{2}} e^{\frac{-b^2}{2T}} + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} T^{-\frac{3}{2}} e^{-\frac{b^2}{2T}} \text{. which is the density of } \tau_b.$$

$$E(\tau_b) = \int_0^\infty P(\tau_b > T)\, dT \quad \text{as } \tau_b \geq 0$$

$$= \int_0^\infty \int_0^{\frac{b}{\sqrt{2T}}} \frac{1}{\sqrt{\pi}} e^{-\hat{x}^2} d\hat{x}\, dT = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^{\frac{b^2}{2\hat{x}^2}} e^{-\hat{x}^2} dT\, d\hat{x}$$

$$= \frac{b^2}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\hat{x}^2}}{\hat{x}^2} d\hat{x} \geq \frac{b^2}{2\sqrt{\pi}} \int_0^1 \frac{1}{\hat{x}^2} d\hat{x} \quad \text{as } \begin{cases} \frac{e^{-\hat{x}^2}}{\hat{x}^2} \geq 0 \text{ on } [1, +\infty) \\ e^{-\hat{x}^2} \leq 1 \text{ on } [0, 1] \end{cases}$$

$$\geq \infty$$

$$\Rightarrow E(\tau_b) = +\infty$$

$\tau := \inf\{t : X_t > a\}$

#20. Given $\nu := \inf\{t : X_t \geq a\}$ and $(X_t)_{t\geq 0}$ is continuous w/ filtration $(\mathcal{F}_t)_{t\geq 0}$

WTS: ν is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ while τ is one w.r.t. $(\mathcal{F}_{t+})_{t\geq 0}$.

Proof:

① Since $\nu = \inf\{t : X_t \geq a\}$, $\{\nu \leq t\} = \sup_{[0,t]}\{X_s \geq a\}$.

Now, by the continuity of $(X_t)_{t\geq 0}$, we have:

$\sup_{[0,t]} X_s(\omega) \geq a \Leftrightarrow \exists (s_i)_{i=1}^{\infty} \subset [0,T) \cap \mathbb{Q}$ s.t. $\lim_{i\to\infty} X_{s_i}(\omega) \geq a$.

$\Rightarrow \sup_{[0,t]}\{X_s \geq a\} = \sup_{[0,t)\cap\mathbb{Q}}\{X_s \geq a\} = \bigcup_{[0,t)\cap\mathbb{Q}}\{X_s \geq a\}$.

Also, $\sup_{[0,t]} X_s(\omega) \geq a \Leftrightarrow \forall n \in \mathbb{N}, \exists m$ s.t. $X_{s_m}(\omega) \geq a - \frac{1}{n}$

$\Rightarrow \sup_{[0,t]}\{X_s \geq a\} = \bigcap_{n=1}^{\infty} \bigcup_{[0,t)\cap\mathbb{Q}}\{X_s \geq a - \frac{1}{n}\}$.

Since, $\forall s \in [0,t)\cap\mathbb{Q}$, $\{X_s \geq a - \frac{1}{n}\} \in \mathcal{F}_{s+} \subset \mathcal{F}_t$, and

$[0,t)\cap\mathbb{Q}$ is countable, $\sup_{[0,t]}\{X_s \geq a\} \in \mathcal{F}_t$

$\Rightarrow \{\nu \leq t\} \in \mathcal{F}_t \Rightarrow \nu$ is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

② Since $\tau = \inf\{t : X_t > a\}$, $\{\tau \leq t\} = \sup_{[0,t]}\{X_s > a\}$.

By continuity of $(X_t)_{t\geq 0}$, Pick n large. $\sup_{[0,t]} X_s(\omega) > a$ iff

$\exists m \in \mathbb{N}$, s.t. $\sup_{[0,t+\frac{1}{m})\cap\mathbb{Q}} X_s(\omega) \geq a + \left(\frac{1}{n} \wedge \left|\sup_{[0,t]} X_s(\omega) - a\right|\right)$

$\Rightarrow \sup_{[0,t]}\{X_s > a\} = \bigcap_{m=1}^{\infty} \sup_{[0,t+\frac{1}{m})\cap\mathbb{Q}}\{X_s \geq a\}$.

$\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$

Since $\sup_{[0,t+\frac{1}{m})\cap\mathbb{Q}}\{X_s > a\} \in \mathcal{F}_{t+\frac{1}{m}}$, $\forall m$, $\bigcap_{m=1}^{\infty} \sup_{[0,t+\frac{1}{m})\cap\mathbb{Q}}\{X_s \geq a\} \in \bigcap_{m=1}^{\infty} \mathcal{F}_{t+\frac{1}{m}}$ // monotonicity

$\Rightarrow \sup_{[0,t]}\{X_s > a\} = \{\tau \leq t\} \in \mathcal{F}_{t+}$.

$\square$

#21. WTS: Given $\tau_1 \leq \cdots \leq \cdots$ are stopping times w.r.t. $(\mathcal{F}_t)$, then $\tau := \lim_{n\to\infty} \tau_n$ is also a stopping time w.r.t. $(\mathcal{F}_t)$

Proof: Since $\tau_1 \cdots \tau_n$ stopping time, for fixed t, we have.

$\{\tau_n \leq t\} \in \mathcal{F}_t$ for $\forall n \in \mathbb{N}$,

Since $\tau_1 \leq \tau_2 \leq \cdots$, we have: $\{\tau_n \leq t\} \supset \{\tau_{n+1} \leq t\}$, $\forall n \in \mathbb{N}$

$\Rightarrow \bigcap_{i=1}^{n} \tau_i^{-1}((-\infty, t]) = \tau_n^{-1}((-\infty, t])$. $(*)$

($\subset$) Now, $\tau^{-1}((-\infty, t])$

$= (\lim_{n\to\infty} \tau_n)^{-1}((-\infty, t]) = (\sup \tau_n)^{-1}((-\infty, t])$

Since $\sup \tau_n \geq \tau_n$, $\forall n \in \mathbb{N}$, $(\sup \tau_n)^{-1}((-\infty, t]) \subset \tau_n^{-1}((-\infty, t])$, $\forall n$

$\Rightarrow (\sup \tau_n)^{-1}((-\infty, t]) \subset \bigcap_{n=1}^{\infty} \tau_n^{-1}((-\infty, t])$

$\Rightarrow \tau^{-1}((-\infty, t]) \subset \bigcap_{n=1}^{\infty} \tau_n^{-1}((-\infty, t])$.

($\supset$) Also, $\forall n \in \mathbb{N}$, $\exists N$ s.t. $\forall m_n \geq N$, we have:

$\tau^{-1}((-\infty, t]) \supset \tau_{m_n}^{-1}((-\infty, t+\frac{1}{n})) = \bigcap_{i=1}^{m_n} \tau_i^{-1}((-\infty, t+\frac{1}{n}])$ by $(*)$

Let $n \to \infty$, we have

$\tau^{-1}((-\infty, t]) \supset \bigcap_{i=1}^{\infty} \tau_i^{-1}((-\infty, t])$

By ($\subset$) & ($\supset$), we have that $\tau^{-1}((-\infty, t]) = \bigcap_{i=1}^{\infty} \tau_i^{-1}((-\infty, t]) \in \mathcal{F}_t$.

Since our choice of t is arbitrary, τ is a stopping time w.r.t. $\mathcal{F}_t$.

$\square$

#22. WTS: $\mathcal{F}_\tau := \{ A : A \cap \{\tau \le t\} \in \mathcal{F}_t \}$ for filtration $(\mathcal{F}_t)$ and stopping time τ is a σ-algebra.

Proof: ①. Since τ is a stopping time w.r.t. $(\mathcal{F}_t)$, we have:

$$\Omega \cap \{\tau \le t\} = \{\tau \le t\} \in \mathcal{F}_t \Rightarrow \Omega \in \mathcal{F}_\tau$$

②. If $A \in \mathcal{F}_\tau \Rightarrow A \cap \{\tau \le t\} \in \mathcal{F}_t$.

Since $\{\tau \le t\} \in \mathcal{F}_t$, $\mathcal{F}_t$ is a σ-algebra,

$$\{\tau \le t\} \cap (A \cap \{\tau \le t\})^c = A^c \cap \{\tau \le t\} \in \mathcal{F}_t$$

$$\Rightarrow A^c \in \mathcal{F}_\tau$$

③. If $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{F}_\tau$, $\Rightarrow A_i \cap \{\tau \le t\} \in \mathcal{F}_t$, $\forall i \in \mathbb{N}$

Since $\bigcup_{i=1}^\infty A_i \supset A_i$, $\forall i$, $(\bigcup_{i=1}^\infty A_i) \cap \{\tau \le t\} \supset A_i \cap \{\tau \le t\}$, $\forall i$

$$\Rightarrow (\bigcup_{i=1}^\infty A_i) \cap \{\tau \le t\} \supset \bigcup_{i=1}^\infty (A_i \cap \{\tau \le t\})$$

Since $\forall N \in \mathbb{N}$, $(\bigcup_{i=1}^N A_i) \cap \{\tau \le t\} = \bigcup_{i=1}^N (A_i \cap \{\tau \le t\}) \subset \bigcup_{i=1}^\infty (A_i \cap \{\tau \le t\})$

$$(\bigcup_{i=1}^\infty A_i) \cap \{\tau \le t\} \subset \bigcup_{i=1}^\infty (A_i \cap \{\tau \le t\})$$

$$\Rightarrow (\bigcup_{i=1}^\infty A_i) \cap \{\tau \le t\} = \bigcup_{i=1}^\infty (A_i \cap \{\tau \le t\})$$

Since $\mathcal{F}_t$ is a σ-algebra, $\bigcup_{i=1}^\infty (A_i \cap \{\tau \le t\}) \in \mathcal{F}_t$

$$\Rightarrow (\bigcup_{i=1}^\infty A_i) \cap \{\tau \le t\} \in \mathcal{F}_t$$

$$\Rightarrow \bigcup_{i=1}^\infty A_i \in \mathcal{F}_\tau$$

By ①, ②, ③, $\mathcal{F}_\tau$ is a σ-algebra. □

#23. An E.g. s.t. τ is not a stopping time and $\mathcal{F}_\tau$ is not a σ-algebra.

Let $(W_t)_{t\in\mathbb{R}^+\cup\{0\}}$ be standard Wiener process.

$\mathcal{F}_t^W := \sigma(W_s : s\in[0,t])$.

$\tau := \inf\{t>0,\ W_{t+1}-W_t \leq 0\}$.

Since $\{\tau \leq t\} = \{W_{s+1}-W_s \geq 0,\ \forall s\in[0,t]\} \notin \mathcal{F}_t$.

as: $W_{t+1}-W_t \perp \mathcal{F}_t$

$\Rightarrow$ τ is not a stopping time.

Since $\Omega\cap\{\tau\leq t\} = \{\tau\leq t\} \notin \mathcal{F}_t \Rightarrow \Omega\notin\mathcal{F}_\tau$

$\Rightarrow$ $\mathcal{F}_\tau$ is not a σ-algebra.

#24. ① WTS: $\forall n \in \mathbb{N}$, τ_n is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$,

given $\tau_n := \frac{[2^n \tau]+1}{2^n}$, and τ a stopping time w.r.t. $(\mathcal{F}_{t+})_{t\geq 0}$.

Proof: Since τ is a stopping time w.r.t. $(\mathcal{F}_{t+})$, we have:

$\forall n \in \mathbb{N}$, $\{\tau \leq t - \frac{1}{n}\} \in \mathcal{F}_{(t-\frac{1}{n})+} \subset \mathcal{F}_t$. as $\exists\, \varepsilon < \frac{1}{n}$

(*) $\Rightarrow \{\tau < t\} = \bigcup_{n=1}^{\infty} \{\tau \leq t - \frac{1}{n}\} \in \mathcal{F}_t$ as $\mathcal{F}_t$ is a σ-algebra.

Let $t \in \mathbb{R}$ be arbitrary, $n \in \mathbb{N}$ be arbitrary,

$\exists\, k$ s.t. $t \in [\frac{k-1}{2^n}, \frac{k}{2^n})$, then we have:

$\{\tau_n \leq t\} = \{\tau_n \leq \frac{k-1}{2^n}\} = \{\tau < \frac{k-1}{2^n}\} \in \mathcal{F}_{\frac{k-1}{2^n}} \subset \mathcal{F}_t$ by (*)

Since our choices of $t \in \mathbb{R}$, $n \in \mathbb{N}$ are arbitrary, we have.

$\forall n \in \mathbb{N}$, τ_n is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$

$\square$

②. WTS: $\mathcal{F}_{\tau_n} \supset \mathcal{F}_{\tau+}$

Proof: Let $A \in \mathcal{F}_{\tau+}$ be arbitrary, then $A \cap \{\tau \leq t\} \in \mathcal{F}_{t+}$.

$A \cap \{\tau < t\} = A \cap \left(\bigcup_{n=1}^{\infty} \{\tau \leq t - \frac{1}{n}\}\right) = \bigcup_{n=1}^{\infty} \left(A \cap \{\tau \leq t - \frac{1}{n}\}\right)$

Since, $\forall n \in \mathbb{N}$, $A \cap \{\tau \leq t - \frac{1}{n}\} \in \mathcal{F}_{(t-\frac{1}{n})+} \subset \mathcal{F}_t$, we have:

$A \cap \{\tau < t\} = \bigcup_{n=1}^{\infty} (A \cap \{\tau \leq t - \frac{1}{n}\}) \in \mathcal{F}_t$ as $\mathcal{F}_t$ is a σ-algebra.

Now, $A \cap \{\tau_n \leq t\} = A \cap \{\tau < \frac{k-1}{2^n}\} \in \mathcal{F}_t$, $\forall n \in \mathbb{N}$

$\Rightarrow A \in \mathcal{F}_{\tau_n}$

$\square$

③. WTS: $\tau_n \downarrow \tau$

Proof: Since $\forall t \in \mathbb{R}^+$, we have:

$\{\tau_n \le t\} = \{\tau < \frac{k_n-1}{2^n}\} \subset \{\tau \le t\} \iff \tau_n \ge \tau$, $\forall n \in \mathbb{N}$

$\{\tau_n \le t\} \subset \{\tau_{n+1} \le t\} \iff \tau_n \ge \tau_{n+1}$, $\forall n \in \mathbb{N}$

Now, $\{\tau \le t\} = \bigcup_{n=1}^{\infty} \{\tau \le t - \frac{1}{n}\}$

But for $\forall n \in \mathbb{N}$, $\exists N$ s.t. $\forall m \ge N$

$$t - \frac{1}{n} < \frac{k_m - 1}{2^m}, \quad k_m := \inf\{k \in \mathbb{N} : \frac{k}{2^m} > t\}$$

$\Rightarrow \bigcup_{n=1}^{\infty} \{\tau \le t - \frac{1}{n}\} \subset \bigcup_{n=1}^{\infty} \{\tau < \frac{k_n-1}{2^n}\} \in \{\lim_{n\to\infty} \tau_n \le t\}$

$\Rightarrow \{\tau \le t\} \subset \bigcup_{n=1}^{\infty} \{\tau \le t - \frac{1}{n}\} \subset \{\lim_{n\to\infty} \tau_n \le t\} \subset \{\tau \le t\}$

as $\lim_{n\to\infty} \tau_n \ge \tau \Rightarrow \{\lim_{n\to\infty} \tau_n \le t\} \subset \{\tau \le t\}$

$\Rightarrow \{\tau \le t\} = \{\lim_{n\to\infty} \tau_n \le t\}$, $\forall t$.

$\Rightarrow \lim_{n\to\infty} \tau_n = \tau$, $(\tau_n \downarrow \tau)$.

$\square$