Limits Theorems II HW #1. Shizhen Xu

#1. Given $\{X_n\}_{n\in\mathbb{N}}$ i.i.d R.V.'s, and $S_n := \sum_{i=1}^{n} X_i$,

$N_t := \sup\{n\in\mathbb{N} : S_n \le t\}$ on $(\Omega, \mathcal{F}, P)$.

WTS: $(N_t)_{t\in\mathbb{R}_+}$ is a stochastic process.

Proof: Pick an arbitrary $t\in\mathbb{R}_+$, we have.

$$N_t = \sup\{n\in\mathbb{N} : S_n \le t\}$$

$$\Rightarrow \quad N_t^{-1}(\{i : i \ge n\}) = S_n^{-1}(\{(t, +\infty)\}), \quad n\in\mathbb{N}$$

Since for each $n\in\mathbb{N}$, $\sum_{i=1}^{n} X_i = S_n$ is $\bigotimes_{i\in\mathbb{N}} \mathcal{F}_i / \mathcal{B}(\mathbb{R})$ measurable $\uparrow \mathcal{F}_i = \mathcal{F},\ \forall i\in\mathbb{N}$

as X_i's are i.i.d R.V.'s, we have.

$$S_n^{-1}(\{(t, +\infty)\}) \in \bigotimes_{i\in\mathbb{N}} \mathcal{F}_i \quad \text{for } \forall n.$$

$$\Rightarrow \quad N_t^{-1}(\{i : i > n\}) \in \bigotimes_{i\in\mathbb{N}} \mathcal{F}_i \quad \text{for } \forall n$$

$$\Rightarrow \quad N_t \text{ is } \bigotimes_{i\in\mathbb{N}} \mathcal{F}_i / 2^{\mathbb{N}} \text{ measurable.}$$

Since the pick of $t\in\mathbb{R}_+$ is arbitrary, we're done.

$\square$

#2. WTS: $\forall B \in \mathcal{B}(\mathbb{R}^d)$ is regular.

Proof:

Let $\mathcal{Y} := \{ B \subset \mathbb{R}^d : \exists U$ open, K compact. $\mu(U \setminus K) < \varepsilon$, $K \subset B \subset U$, for $\forall \varepsilon > 0$, $\forall \mu$ probability measure $\}$.

Claim: $\mathcal{Y}$ is a σ-algebra

①: $\phi \in \mathcal{Y}$ as ϕ is both compact & open

②: If $A \in \mathcal{Y}$, for arbitrary $\varepsilon > 0$, μ; $\exists U, K$ s.t. $\mu(U \setminus K) < \varepsilon$

K compact $\Rightarrow$ K^c open

U open $\Rightarrow$ U^c closed in $\mathbb{R}^d$ $\Rightarrow \exists \{\widehat{K_i}\}_{i=1}^{\infty}$ compact s.t.

$$\begin{cases} \bigcup_{i=1}^{\infty} \widehat{K_i} = U^c \\ \mu\left(\bigcup_{i=1}^{\infty} \widehat{K_i} \setminus \bigcup_{i=1}^{N} \widehat{K_i}\right) < \frac{\varepsilon}{2} \quad \text{for some } N < \infty \end{cases}$$

$$\Rightarrow \mu\left(K^c \setminus \bigcup_{i=1}^{N} \widehat{K_i}\right) \leq \mu\left(K^c \setminus \bigcup_{i=1}^{\infty} \widehat{K_i}\right) + \mu\left(\bigcup_{i=1}^{\infty} \widehat{K_i} \setminus \bigcup_{i=1}^{N} \widehat{K_i}\right)$$

$$\leq \varepsilon/2 + \varepsilon/2 < \varepsilon$$

and $K^c \supset A^c \supset \bigcup_{i=1}^{N} \widehat{K_i}$, K^c open, $\bigcup_{i=1}^{N} \widehat{K_i}$ compact

$\Rightarrow A^c \in \mathcal{Y}$

③: If $\{A_i\}_{i=1}^{\infty} \subset \mathcal{Y}$, for fixed $\varepsilon > 0$, μ, $\exists \{U_i\}_{i=1}^{\infty}$, $\{K\}_{i=1}^{\infty}$ s.t.

$\mu(U_i \setminus K_i) < \frac{\varepsilon}{2^{i+2}}$, and $U_i \supset A_i \supset K_i$ for $\forall i \in \mathbb{N}$.

$\Rightarrow U := \bigcup_{i=1}^{\infty} U_i$, $\exists N$ s.t. $\mu\left(\bigcup_{i=1}^{\infty} K_i \setminus \bigcup_{i=1}^{N} K_i\right) < \frac{\varepsilon}{2}$ as μ is finite.

$\Rightarrow \mu\left(U \setminus \bigcup_{i=1}^{N} K_i\right) \leq \sum_{i=1}^{\infty} \mu(U_i \setminus K_i) + \mu\left(\bigcup_{i=1}^{\infty} K_i \setminus \bigcup_{i=1}^{N} K\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{Y}$

By ①, ②, ③, $\mathcal{F}$ is a σ-algebra.

Now, fix a closed $F \subset \mathbb{R}^d$, define $G_n := \{y : |y - x| < \frac{1}{n},\ x \in K\}$

Also, fix a point $x' \in F$, define $K_n := F \cap \overline{B}_n(x)$

$\Rightarrow$ for arbitrary $\varepsilon > 0$, μ

$\exists N < \infty$ s.t. $\mu(G_N \setminus \bigcup_{i=1}^{N} K_i) < \varepsilon$

$\Rightarrow$ Since $\varepsilon > 0$ and μ are arbitrary, $F \in \mathcal{F}$

Since $F \subset \mathbb{R}^d$ closed is arbitrary, $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{F}$.

$\Rightarrow$ $\forall B \in \mathcal{B}(\mathbb{R}^d)$ is regular.

$\square$

#3. Given $C_{t_1 \cdots t_n}(B) := \{ x \in \mathbb{R}^T : (x_{t_1}, \cdots, x_{t_n}) \in B \}$

WTS: $\{ C_{t_1 \cdots t_n}(B) : n < \infty, B \in \mathcal{B}(\mathbb{R}^n) \} =: \mathcal{G}$ is an algebra.

①. $\mathcal{B}(\mathbb{R}^n)$ is a σ-algebra $\Rightarrow \mathbb{R}^n \in \mathcal{B}\mathbb{R}^n$

$\mathbb{R}^T = \{ x \in \mathbb{R}^T : (x_{t_1}, \cdots x_{t_n}) \in \mathbb{R}^n \} \in \mathcal{G}$.

② If $A \in \mathcal{G}$, then $A = \{ x \in \mathbb{R}^T : (x_{t_1}, \cdots x_{t_n}) \in \hat{A} \}$ for some $(t_1, \cdots, t_n)$ and $\hat{A} \in \mathcal{B}(\mathbb{R}^n)$.

$\Rightarrow \hat{A}^C \in \mathcal{B}(\mathbb{R}^n) \Rightarrow A^C = \{ x \in \mathbb{R}^T : (x_{t_1} \cdots x_{t_n}) \in \hat{A}^C \} \in \mathcal{G}$.

③. Assume $\{A_i\}_{i=1}^n \subset \mathcal{G}$, then $\exists \{ t_{ij} \}_{j \in \{1, \cdots, k(i)\}}$ and

$\hat{A}_i \in \mathcal{B}(\mathbb{R}^{k(i)})$ for each $i \in \{1, \cdots, n\}$, such that

$A_i = \{ x : (x_{t_{i1}} \cdots x_{t_{ik(i)}}) \in \hat{A}_i \}$

Let $\hat{T} := \bigcap_{i=1}^n \{ t_{i1}, \cdots, t_{ik(i)} \} = \{ t_1, \cdots, t_{c(i)} \}$ for some $c(i) \in \mathbb{Z}$.
$c(i) \leq \bigwedge_{i=1}^n k(i)$

$\Rightarrow \bigcup_{i=1}^n A_i = \{ x \in \mathbb{R}^T : (x_{t_1}, \cdots, x_{t_{c(i)}}) \in \bigcap_{i=1}^n \hat{A}_i \} \in \mathcal{G}$.

(If $\hat{T} = \phi$, $\bigcup_{i=1}^n A_i = \mathbb{R}^T \in \mathcal{G}$)

By ①, ②, ③, $\mathcal{G}$ is an algebra.

$\square$

#4. Given $E := \{ C_{t_1 \cdots t_n}(B_1 \cdots B_n) : n < \infty,\ B_i \in \mathcal{B}(\mathbb{R}),\ \forall i \}$,

$C_{t_1 \cdots t_n}(B_1 \cdots B_n) := \{ x \in \mathbb{R}^T : x_{t_i} \in B_i,\ \forall i \in \{1, \cdots, n\} \}$.

G defined as in #3.

WTS: $\sigma(E) = \sigma(G)$.

$(\subset)$: Since $\mathcal{B}(\mathbb{R}^n) \supset \{ \prod_{i=1}^n B_i,\ B_i \in \mathcal{B}(\mathbb{R}) \}$ for $\forall n < \infty$,

we have $E \subset G \Rightarrow \sigma(E) \subset \sigma(G)$.

$(\supset)$: $\forall n < \infty$, $\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^n \mathcal{B}(\mathbb{R})$ as $\mathbb{R}$ is separable.

$= \sigma(\{ \prod_{i=1}^n B_i,\ B_i \in \mathcal{B}(\mathbb{R}) \})$

Define $\hat{E} := \{ \prod_{i=1}^\infty B_i : C_{t_1 \cdots t_\infty}(\prod_{i=1}^\infty B_i) \in \sigma(E) \}$.

Claim: $\hat{E}$ is a σ-algebra as ①. $\mathbb{R}^\infty \in \hat{E}$

②. If $A \in \hat{E}$, $A^c = \prod_{i=1}^\infty A_i^c$

Since $C_{t_1 \cdots t_\infty}(\prod_{i=1}^\infty A_i^c) \in \sigma(E)$, $A^c \in \hat{E}$

③. If $\{A_i\}_{i=1}^\infty \in \hat{E}$, $A_i = \prod_{j=1}^\infty A_{ij}$ for $\forall i \in \mathbb{N}$

Since $\bigcup_{i=1}^\infty C_{t_{i1} \cdots t_{i\infty}}(\prod_{j=1}^\infty A_{ij}) \in \sigma(E)$, $\bigcup_{i=1}^\infty A_i \in \hat{E}$

$\Rightarrow$ $\hat{E}$ is a σ-algebra, $\hat{E} \supset \bigotimes_{i=1}^n \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^n)$

$\Rightarrow \sigma(E) \supset G \Rightarrow \sigma(E) \supset \sigma(G)$

By $(\supset)$, $(\subset)$, we're done

$\square$

#5. Define $\mathcal{G}_T := \sigma\{C(t_1, \cdots t_n, B) : t_1, \cdots t_n \in T\}$, $T \subset \Pi$

WTS: $\mathcal{B}(\mathbb{R}^\Pi) = \bigcup_{\substack{T \subset \Pi \\ \text{countable}}} \mathcal{G}_T$.

($\subset$): Claim: $\bigcup_{\substack{T \subset \Pi \\ \text{countable}}} \mathcal{G}_T$ is a σ-algebra

①. Since $\forall T \subset \Pi$, $\mathcal{G}_T$ is σ-algebra $\Rightarrow \mathbb{R}^\Pi \in \mathcal{G}_T$

$\Rightarrow \mathbb{R}^\Pi \in \bigcup_{\substack{T \subset \Pi \\ \text{countable}}} \mathcal{G}_T$ (denoted by $\bigcup_T \mathcal{G}_T$ below)

②. If $A \in \bigcup_T \mathcal{G}_T$, then $A \in \mathcal{G}_T$ for some T

$\Rightarrow A^c \in \mathcal{G}_T \Rightarrow A^c \in \bigcup_T \mathcal{G}_T$

③. If $\{A_i\}_{i=1}^\infty \in \bigcup_T \mathcal{G}_T$, then $A_i \in \mathcal{G}_{T_i}$ for each i.

Let $\hat{T} = \bigcup_{i=1}^\infty T_i$, $\bigcup_{i=1}^\infty A_i \in \mathcal{G}_{\hat{T}} \subset \bigcup_T \mathcal{G}_T$ as $\hat{T}$ is countable.

$\Rightarrow \bigcup_T \mathcal{G}_T$ is a σ-algebra

Since $\mathcal{B}(\mathbb{R}^\Pi) = \sigma(\{C(t_1, \cdots t_n, B),\ n \in \mathbb{N},\ B \in \mathcal{B}(\mathbb{R}^n)\})$

and $\bigcup_T \mathcal{G}_T \supset \{C(t_1 \cdots t_n, B),\ n \in \mathbb{N},\ B \in \mathcal{B}(\mathbb{R}^n)\}$

$\Rightarrow \mathcal{B}(\mathbb{R}^\Pi) \subset \bigcup_T \mathcal{G}_T$

($\supset$): Let $T \subset \Pi$ countable be arbitrary, then $\exists$ $\begin{cases} \{T_n\}_{n=1}^\infty \text{ s.t. } \bigcup_{n=1}^\infty T_n = T \\ \{B_n\}_{n=1}^\infty \text{ s.t. } \bigcap_{n=1}^\infty B_n = B. \end{cases}$

$\Rightarrow C(T, B) = \bigcap_{n=1}^\infty C(T_n, B_n)$

$\Rightarrow \mathcal{B}(\mathbb{R}^\Pi) \supset C(T, B) \Rightarrow \mathcal{B}(\mathbb{R}^\Pi) \supset \mathcal{G}_T$

Since $T \subset \Pi$ countable is arbitrary, $\mathcal{B}(\mathbb{R}^\Pi) \supset \bigcup_T \mathcal{G}_T$

By ($\subset$), ($\supset$), we're done. $\square$

#6. WTS: $\exists$. $(W)_{t\in\pi}$ using characteristic functions.

Proof:

①. Define $H_k^{(n)}(t) := \begin{cases} 2^{(n-1)/2} & \frac{k-1}{2^n} \le t < \frac{k}{2^n} \\ -2^{(n-1)/2} & \frac{k}{2^n} \le t < \frac{k+1}{2^n} \\ 0 & \text{otherwise.} \end{cases}$

$S_k^{(n)}(t) := \int_0^t H_k^{(n)}(u)\,du$, $t \in [0,1]$, $n \ge 0$, $k \in I(n)$.

$I(n) = \{2k+1,\ 0 \le k \le \frac{2^n-1}{2}\}$.

$B_t^{(n)} := \sum_{m=0}^{n} \sum_{k\in I(m)} \xi_k^{(m)}(w) S_k^{(m)}(t)$, where $\{\xi_k^{(n)}\}_{n\in\{0,\dots\},\, k\in I(n)}$ is defined as i.i.d. standard normal.

②. Claim: $B_t^{n}(w) \xrightarrow[n\to\infty]{} B_t(w)$ uniformly in t for a.e. $w \in \Omega$.

Proof: defined $b_n := \max_{k\in I(n)} |\xi_k^{(n)}|$

$\Rightarrow P(|\xi_k^{(n)}| \ge x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2}\,du \le \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{u}{x} e^{-u^2/2} \le \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x}$

$\Rightarrow P(b_n > n) = P\left(\bigcup_{k\in I_n} \{|\xi_k^{(n)}| \ge n\}\right) \le 2^n P(|\xi_k^{(n)}| \ge x) \le \sqrt{\frac{2}{\pi}} \frac{2^n e^{-n^2/2}}{n}$

$\sum_{n=1}^{\infty} \frac{2^n e^{-n^2/2}}{n} < \infty \Rightarrow \exists\ \hat{\Omega}$; $P(\hat{\Omega}) = 1$ s.t. if $w \in \hat{\Omega}$ we have

by Borel Cantelli $\quad \exists\ n(w)$ w/ $b_n(w) \le n$, if $n \ge n(w)$.

$\Rightarrow \sum_{n=n(w)}^{\infty} \sum_{k\in I(n)} |\xi_k^{(n)} S_k^{(n)}| \le \sum_{n=n(w)}^{\infty} n 2^{-\frac{1}{2}(n+1)} < \infty$

Since absolute convergence implies convergence on $\mathbb{R}^\pi$,

$B_t^{(n)}(w) \xrightarrow[n\to\infty]{} B_t(w)$, if $w \in \hat{\Omega}$. for some B_t

Since $B_t^{(n)}(w) \subset C^{(0)}(\pi) \Rightarrow B_t(w) \in C^{(0)}(\pi)$ by uniform convergence

③. Claim: B_t defined in ② is Wiener process on $[0,1]$.

Proof: Pick $n \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n \le 1$

Pick $\{\lambda_i\}_{i=1}^{n} \subset \mathbb{R}$ w/ $\lambda_{n+1} := 0$

Since $\{\xi_k^{(m)}\}$'s are i.i.d. standard normal,

$B_t^{(N)}(\omega) = \sum_{m=0}^{n} \sum_{k \in I(m)} \xi_k^{(m)}(\omega) S_k^{(m)}(t)$ implies that

$$E\left[\exp\left(i\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j) B_{t_j}^{(N)}\right)\right] = E\left[\exp\left(-i\sum_{m=0}^{n}\sum_{k\in I(m)} \xi_k^{(m)} \sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j) S_k^{(m)}(t_j)\right)\right]$$

$$= \prod_{m=0}^{N}\prod_{k\in I(m)} \exp\left[-\frac{1}{2}\left(\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j) S_k^{(m)}(t_j)\right)\right]$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}(\lambda_{j+1}-\lambda_j)(\lambda_{i+1}-\lambda_i)\ K(N)\right]$$

where $K(N) := \sum_{m=0}^{N}\sum_{k\in I(m)} S_k^{(m)}(t_i) S_k^{(m)}(t_j) \underset{N\to\infty}{\longrightarrow} t_i \wedge t_j$

$$\overset{N\to\infty}{\Rightarrow} E\left[\exp\left(i\sum_{j=1}^{n}\lambda_j(B_{t_j}-B_{t_{j-1}})\right)\right] = E\left[\exp\left(-i\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j) B_{t_j}\right)\right]$$

$$= \exp\left[-\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}(\lambda_{j+1}-\lambda_j)(\lambda_{i+1}-\lambda_i)t_j - \frac{1}{2}\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)^2 t_j\right]$$

$$= \exp\left[-\sum_{j=1}^{n-1}(\lambda_{j+1}-\lambda_j)(-\lambda_{j+1})t_j - \frac{1}{2}\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)^2 t_j\right]$$

$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n-1}(\lambda_{j+1}^2-\lambda_j^2)t_j - \frac{1}{2}\lambda_n^2 t_n\right]$$

$$= \prod_{j=1}^{n} \exp\left[-\frac{1}{2}\lambda_j^2(t_j - t_{j-1})\right]$$

Since characteristic functions determine distribution, we have

$B_{t_j} - B_{t_{j-1}} \sim N(0, t_j - t_{j-1})$ independent $\Rightarrow$ B is Wiener on $[0,1]$

Since we can find B_t on $[k, k+1]$, $\forall k \in \mathbb{N}$, we're done

$\square$

#7. WTS: $(X_t)_{t\in[0,1]}$ uncountable family of iid R.V.'s w/ non-degenerate distribution, then $(X_t)_{t\in[0,1]}$ has no continuous modification.

Proof: (By contradiction) $(X_t)_{t\in[0,1]}$ is an uncountable family,

$\exists\ t \in [0,1]$ s.t. $\forall \varepsilon > 0$ $\exists\ s \in (t-\varepsilon, t+\varepsilon)$ and $X_s \in (X_t)_{t\in[0,1]}$

Pick a sequence $\{t_i\}_{i\in\mathbb{N}}$ from $\{t\in[0,1] : X_t \in (X_t)_{t\in[0,1]}\} =: \pi$

s.t. $\lim_{i\to\infty} t_i = t$, $t_i \neq t$, $\forall i \in \mathbb{N}$.

Since $(X_t)_{t\in[0,1]}$ are iid w/ non-degenerate distribution

$\Rightarrow$ $Var(X_t) = \sigma^2 \neq 0$ for $\forall t \in \pi$.

$\Rightarrow$ $\lim_{i\to\infty} \| X_{t_i} - X_t \|_{L^2(P)} = 2\sigma^2 \neq 0$ by iid

Now, assume $\exists\ \hat{X}$ s.t. $\begin{cases} P(\hat{X}_t \neq X_t) = 0, & \forall t \in \pi \\ |\hat{X}_t - \hat{X}_s| < C|t-s| & a.s. \end{cases}$

Let $N := \bigcup_{i=1}^{\infty} \{\hat{X}_{t_i} \neq X_{t_i}\} \cup \{\hat{X}_t \neq X_t\}$ w/ $P(N) = 0$.

$\Rightarrow$ $\lim_{i\to\infty} \hat{X}_{t_i} = \hat{X}_t$ P-a.s.

$\Rightarrow$ $E \lim_{i\to\infty} |\hat{X}_{t_i} - \hat{X}_t|^2 = 0$

But $E \lim_{i\to\infty} |\hat{X}_{t_i} - \hat{X}_t|^2 = \lim_{i\to\infty} E|\hat{X}_{t_i} - \hat{X}_t|^2$ apply D.C.T. as $|\hat{X}_{t_i} - \hat{X}_t|^2 \leq C|t_i - t|^2$

$= \lim_{i\to\infty} E|X_{t_i} - X_t|^2$

$= 2\sigma^2 \neq 0$ $\rightarrow\leftarrow$ (contradiction)

$\square$

#8. WTS: If $X : [0,1]^d \times \Omega \to \mathbb{R}$ satisfies $E|X_s - X_t|^\alpha \le C|s-t|^{d+\beta}$ for $\alpha, \beta, C > 0$, $\forall t, s \in [0,1]^d$, then $\exists$ a continuous modification of X on $[0,1]^d$.

Proof:

① Define $<$ s.t. $(s_1, \dots, s_d) < (t_1, \dots, t_d) \Leftrightarrow s_i < t_i, \forall i \in \{1, \dots, d\}$

Let $L_n := \left\{ \frac{k}{2^n} : k \in \{0, \dots, 2^n - 1\} \right\}^d$, $n \ge 1$, $L := \bigcup_{n=1}^{\infty} L_n$

Define $N_n(s) := \{ t \in L_n : s < t, \|t - s\|_{\ell^2} = 2^{-n} \}$, $s \in L_n$

②. For $s \in L_n$, $t \in N_n(s)$, by Chebyshev's Ineq: $\forall \gamma \in (0 \quad)$.

$$P(|X_t - X_s| \ge 2^{-\gamma n}) \le \frac{E|X_t - X_s|^\alpha}{(2^{-\gamma n})^\alpha} \le C 2^{-n(d+\beta-\alpha\gamma)}$$

as $|t-s| \le 2^{-n}$ and $E|X_t - X_s|^\alpha \le C|t-s|^{d+\beta}$.

$$\Rightarrow P\left(\max_{\substack{s \in L_n \\ t \in N_n(s)}} |X_t - X_s| \ge 2^{-\gamma n} \right) \le d\, C 2^{-n(\beta - \alpha\gamma)} \quad \text{as } |N_n(s)| = d.$$

$\sum_{n=1}^{\infty} 2^{-n(\beta-\alpha\gamma)} < \infty \Rightarrow \exists \hat{\Omega}$ w/ $P(\hat{\Omega}) = 1$ s.t. $\forall w \in \hat{\Omega}$ $\exists N(w)$ s.t. $\max_{\substack{s \in L_n \\ t \in N_n(s)}} |X_t(w) - X_s(w)| < 2^{-\gamma n}$, for $\forall n \ge N(w)$

③. Define $R_n(s) := \{ t \in L_n : s < t, \|t - s\|_{\infty} = 2^{-n} \}$

$\Rightarrow$ for $s \in L_n$, $t \in R_n$, $\exists$ $s = s_0, s_1, \dots, s_m = t$ w/ $m \le d$ and $s_i \in N_n(s_{i-1})$, $i \in \{1, \dots, m\}$.

(*) $\Rightarrow \max_{\substack{s \in L_n \\ t \in R_n(s)}} |X_t(w) - X_s(w)| \le d\, 2^{-\gamma n}$, $\forall n \ge N(w)$, $w \in \hat{\Omega}$

④. For fixed $\omega \in \hat{\Omega}_0$, $n \geq N(\omega)$, from (*) we obtain

$$|X_t(\omega) - X_s(\omega)| \leq 2d \sum_{j=n}^{n+1} 2^{-\gamma j}$$

Now, assume, $|X_t(\omega) - X_s(\omega)| \leq 2d \sum_{j=n+1}^{m-1} 2^{-\gamma j}$, $\forall m \in \{n+1, \cdots, M-1\}$.

Let $t, s \in L_M$, $s < t$. $\exists \begin{cases} \tilde{s} \in L_{M-1} \cap R_M(s) \\ \tilde{t} \in L_{M-1} \text{ w/ } t \in R_M(\tilde{t}) \end{cases}$

s.t. $s < \tilde{s} < \tilde{t} < t$

$\Rightarrow$ $|X_{\tilde{s}}(\omega) - X_s(\omega)| \leq d\, 2^{-\gamma M}$ and $|X_{\tilde{t}}(\omega) - X_t(\omega)| \leq d 2^{-\gamma M}$

$\Rightarrow$ $|X_{\tilde{t}}(\omega) - X_{\tilde{s}}(\omega)| \leq 2d \sum_{j=n+1}^{M} 2^{-\gamma j}$

$\Rightarrow$ $\forall$ $s, t \in L$ w/ $s < t$ and $0 < \|t - s\|_\infty < 2^{-N(\omega)}$,

pick $n \geq N(\omega)$ s.t. $2^{-(n+1)} \leq \|t - s\|_\infty < 2^{-n}$

$\Rightarrow$ $|X_t(\omega) - X_s(\omega)| \leq 2d \sum_{j=n}^{\infty} 2^{-\gamma j} \leq \frac{2d}{1 - 2^{-\gamma}} \|t - s\|_\infty^\gamma$

Since $\|\cdot\|_\infty$ & $\|\cdot\|_{\ell^2}$ are equivalent on $\mathbb{R}^d$, we have.

(**) $|X_t(\omega) - X_s(\omega)| \leq C \|t - s\|_{\ell^2}^\gamma$ for some $C \in \mathbb{R}$.

Since our choice of $\omega \in \hat{\Omega}_0$ is arbitrary.

⑤. Define $\hat{X}_t(\omega) := \begin{cases} X_t(\omega) & \text{if } t \in L \\ \lim_{s \to t} X_s(\omega) & \text{if } t \notin L \end{cases}$ (note: L is dense in $\mathbb{R}^d$).

By (**) $\Rightarrow$ $\hat{X}_\cdot(\omega)$ is continuous on $\mathbb{R}^d$, $\forall$ $\omega \in \hat{\Omega}_0$ (a.e.)

$\begin{cases} \text{For } t \in L, \quad \hat{X}_t = X_t \text{ a.s.} & \Rightarrow P(\hat{X}_t \neq X_t) = 0 \\ \text{For } t \in \mathbb{R}^d \setminus L, \ \hat{X}_t = \lim_{\substack{s \to t \\ s \in L}} X_s \text{ a.s.} \end{cases}$

$\square$

#9. WTS: $E|X_t - X_s| \leq C|t-s|$ is not sufficient for $\exists$ continuous modification.

Proof: Let τ s.t. $P(\{\tau \leq t\}) = 1 - e^{-t}$, $t \in \mathbb{R}_+ \cup \{0\}$.

Define $X_t := \mathbb{1}_{\{\tau \leq t\}}$.

$\Rightarrow$ Assume WOLG that $t > s$, $t, s \in \mathbb{R}_+ \cup \{0\}$.

$$E|X_t - X_s| = E\left(\mathbb{1}_{\{\tau \in (s,t]\}}\right) = P(\{\tau \in (s,t]\})$$

$$= (1-e^{-t}) - (1-e^{-s}) = e^{-s}(1-e^{-(t-s)})$$

$$\leq |t-s|$$

Now, if $\hat{X}$ is a modification of $X \Rightarrow \forall t$, $P(\hat{X}_t \neq X_t) = 0$.

$\forall \varepsilon > 0$, $P\{\tau^{-1}(t-\varepsilon, t+\varepsilon)\} > 0$

$\hat{X}_t = X_t$ on $\tau^{-1}(t-\varepsilon, t+\varepsilon) \Rightarrow \{0,1\} \subset \hat{X}_{(t-\varepsilon, t+\varepsilon)}(w)$.

for $\forall w \in \tau^{-1}(t-\varepsilon, t+\varepsilon)$.

$\Rightarrow \hat{X}_t$ is not continuous in t

Since our choice of $\hat{X}_t$ is arbitrary, we are done.

$\square$

#10. WTS: $\exists$ Poisson counting process.

Poisson means a discrete distribution

Let $\{X_i\}_{i\in\mathbb{N}}$ be iid standard poisson : $P_{X_i}(t) = e^{-t}$, $\forall i\in\mathbb{N}$

This is called an exponential density

Define $S_n := \sum_{i=1}^{n} X_i$, $\forall n\in\mathbb{N}$

$$N_t := \max\{n : S_n \le t\}, \ \forall t\in\mathbb{R}_+$$

①. Since X_i's are ~~non-negative~~ (positive) a.s. $\Rightarrow$ S_n is non-decreasing a.s.

$\Rightarrow$ N_t is non-decreasing a.s. by design.

②. $\forall t\in\mathbb{R}_+$, N_t is piece-wise constant a.s

$$N_t \equiv n \text{ on } [S_n, S_{n+1})$$

where $X_{n+1} \ge 0$ a.s. $\Rightarrow$ $[S_n, S_{n+1}) \ne \{S_n\}$ a.s.

$\Rightarrow$ N_t is right continuous.

③. Since $X_i > 0$ a.s., the jumps of N_t has value 1 a.s.

Now, by the memorylessness of X_i's ($P(X>t+x) = P(X>t)\,P(X>x)$) and the independence of X_i's, we have $\forall z$ starts at t

$$P(Z>z \mid N(t)=n, S_n=\tau) = P(X_{n+1} > z+t-\tau \mid X_{n+1} > t-\tau) = P(X_{n+1}>z)$$

Pick $0=t_0<t_1, \cdots < t_n = t \subset \mathbb{R}_+$, we have : (next page)

$$N_z - N_\tau = N_{z-\tau} \Rightarrow \text{Increment stationary.}$$

z is independent of $N_{t_1}, N_{t_2}-N_{t_1}, \cdots, N_{t_n}-N_{t_{n-1}}$

Let $t_{n+1} > t \Rightarrow N_{t_{n+1}} - N_{t_n}$ is independent of $N_{t_1}, \cdots, N_{t_n}-N_{t_{n-1}}$

$\Rightarrow$ Increments are independent.

Also, by construction S_n is poisson w/ $f_{S_n}(t) = \dfrac{t^{n-1}e^{-t}}{(n-1)!}$

$$\sum_{i=n}^{\infty} P_{N_t}(i) = \int_0^t f_{S_n}(\tau)\,d\tau$$

$$P_{N_t}(i) = \frac{t^n e^{-t}}{n!}$$

$\Rightarrow (N_t)_{t\in\mathbb{R}_+}$ is Poisson counting process satisfying ① ② ③

$\square$

#11. WTS: $\hat{X} \in (\mathbb{R}^{\Omega})^2$ is not Gaussian but has marginal distribution that is Gaussian.

Let $\hat{X} := (X, Y)^T$ w/ $P_{XY}(x,y) := \begin{cases} \frac{1}{\pi \sigma_X \sigma_Y} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2}\right)\right) & , xy \geq 0 \\ 0 & , xy \leq 0 \end{cases}$

$$\Rightarrow P_X(x) = \int_{-\infty}^{+\infty} P_{XY}(x,y)\, dy = \int_{-\infty}^{0} P_{XY}(x,y)\, dy\, \mathbb{1}_{\{x \leq 0\}} + \int_{0}^{+\infty} P_{XY}\, dy\, \mathbb{1}_{\{x \geq 0\}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma_X^2}\right)$$

$P_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\right)$ similarly.

$\Rightarrow$ $\hat{X}$ is not Gaussian but has marginal Gaussian distribution.

#12. $Y \sim \mathcal{N}(a, C)$ is a d-dim Gaussian vector, $Z := AY$,

where A is $\mathbb{R}^{n \times d}$,

WTS: Z is a Gaussian vector $\sim \mathcal{N}(z, \tilde{C})$

Proof:

Since $Y \sim \mathcal{N}(a, C)$, we have $\forall\ \vec{\alpha} \in \mathbb{R}^d$, we have

$\vec{\alpha} \cdot Y$ is Gaussian (by #13).

Now, $\forall\ \vec{n} \in \mathbb{R}^d$, we have

$$\vec{n} \cdot Z = \sum_{i=1}^{n} n_i Z_i = \sum_{i=1}^{n} n_i \sum_{j=1}^{d} A_{ij} Y_j = \sum_{j=1}^{d} \left(\sum_{i=1}^{n} n_i A_{ij} \right) Y_j$$

$$= \sum_{j=1}^{d} (A^T \vec{n})_j Y_j = (A^T \vec{n}) \cdot Y, \quad A^T \vec{n} \in \mathbb{R}^d$$

$\Rightarrow$ $\forall\ \vec{n} \in \mathbb{R}^n$, $\vec{n} \cdot Z$ is Gaussian as $(A^T \vec{n}) \cdot Y$ is Gaussian

$\Rightarrow$ Z is Gaussian.

$\square$

$$EY = a \Rightarrow EZ = E(AY) \overset{\text{by linearity}}{=} AEY = Aa$$

$$E(YY^T) = C \Rightarrow E(ZZ^T) = E(AYY^T A^T) \overset{\text{by linearity}}{=} AE(YY^T)A^T$$

$$= ACA^T$$

$$\Rightarrow Z \sim \mathcal{N}(Aa, ACA^T)$$

#13. WTS: X is Gaussian $\Leftrightarrow$ $\forall b \in \mathbb{R}^d$, $\langle b, X\rangle$ is Gaussian.

$(\Rightarrow)$: If X is Gaussian, let $a := EX \in \mathbb{R}^d$, $C := E(X-EX)(X-EX)^T \in \mathbb{R}^{d\times d}$, we have the characteristic function for X:

$$(*) \quad \varphi_X(\lambda) = E e^{i\langle\lambda, X\rangle} = e^{i\langle\lambda, a\rangle - \frac{1}{2}\langle C\lambda, \lambda\rangle} = e^{i\lambda^T a - \frac{1}{2}(\lambda^T C\lambda)}$$

Pick $b \in \mathbb{R}^d$ arbitrarily, we have:

$$\varphi_{\langle b,x\rangle}(\tilde{\lambda}) = E\, e^{i\tilde{\lambda}\langle b,x\rangle} = E e^{i\langle\tilde{\lambda} b, X\rangle}$$

$$= e^{i(\tilde{\lambda} b)^T a - \frac{1}{2}((\tilde{\lambda} b)^T C(\tilde{\lambda} b))} \quad \text{by } (*)$$

$$= e^{i\tilde{\lambda}(b^T a) - \frac{1}{2}\tilde{\lambda}^2(b^T C b)}$$

$$\Rightarrow \langle b, x\rangle \sim \mathcal{N}(b^T a, b^T C b), \quad \forall b \in \mathbb{R}^d$$

$(\Leftarrow)$: If $\forall b \in \mathbb{R}^d$, $\langle b, X\rangle$ is Gaussian, we have:

$$(**) \quad \varphi_{\langle b,x\rangle}(\tilde{\lambda}) = E e^{i\tilde{\lambda}\langle b,x\rangle} = e^{i\tilde{\lambda} a_b - \frac{1}{2}\tilde{\lambda}^2\sigma_b^2}, \text{ where}$$

$$\begin{cases} a_b = E\langle b,x\rangle \\ \sigma_b^2 = E(\langle b,x\rangle - E\langle b,x\rangle)^2 = E\langle b, x-Ex\rangle^2 = E\sum_{i=1}^d b_i(x_i - Ex_i)\sum_{j=1}^d b_j(x_j - Ex_j) \\ \quad = E\, b^T(X-EX)(X-EX)^T b = b^T\left(E(X-EX)(X-EX)^T\right) b \end{cases}$$

$$\Rightarrow \varphi_X(\lambda) = E e^{i\langle\lambda,x\rangle}, \quad \text{let } \lambda = \tilde{\lambda} b, \quad b := \left(\frac{\lambda_1}{\tilde{\lambda}}, \frac{\lambda_2}{\tilde{\lambda}}, \dots, \frac{\lambda_d}{\tilde{\lambda}}\right)^T$$

$$\text{by } (**) \quad = E e^{i\tilde{\lambda}\langle b,x\rangle} = e^{i\tilde{\lambda} E\langle b,X\rangle - \frac{1}{2}\tilde{\lambda}^2 b^T \underbrace{(E(X-EX)(X-EX)^T)}_{\overset{!!}{K}\text{ positive semidefinite}} b}$$

$$\text{where } \begin{cases} \tilde{\lambda} E\langle b,x\rangle = E\langle\tilde{\lambda} b, X\rangle = \langle\lambda, EX\rangle \\ \tilde{\lambda}^2 b^T K b = (\tilde{\lambda} b)^T K(\tilde{\lambda} b) := \lambda^T K\lambda \end{cases}$$

$$\Rightarrow X \sim \mathcal{N}(EX, K)$$

$\square$

#14. WTS: $(\cdot,\cdot): \mathbb{R}_+^2 \to \mathbb{R}_+$ is positive semi-definite.
$(s,t) \to s\wedge t$

Proof: Pick $n\in\mathbb{N}$ arbitrarily, and pick $\{t_1,\cdots,t_n\}\subset\mathbb{R}_+$ arbitrarily.

Let $(t_i,t_j)_{i,j\in\{1,\dots n\}} := (t_i\wedge t_j)_{i,j\in\{1,\dots n\}} \in \mathbb{R}_+^{n\times n}$

Since $t_i\wedge t_j = \langle \mathbb{1}_{[0,t_i]}, \mathbb{1}_{[0,t_j]}\rangle_{L^2(\mathbb{R}_+)}$,

let $I := (\mathbb{1}_{[0,t_1]},\cdots,\mathbb{1}_{[0,t_n]})^T$, we have:

$$(t_i,t_j)_{i,j} = \langle I_i, I_j\rangle_{L^2(\mathbb{R}_+)} =: E(II^T)$$

Now, for $\forall\, d\in\mathbb{R}^n$, we have:

$$d^T(t_i,t_j)_{i,j}\, d = d^T E(II^T)\, d = E(d^T I I^T d)$$
$$= E\,(d^T I)^2 \geq 0$$

$\Rightarrow (t_i,t_j)_{i,j}$ is positive semi-definite $n\times n$ matrix

Since our choice of n, $(t_1\cdots t_n)\subset\mathbb{R}_+$ are arbitrary,
we conclude $(\cdot,\cdot): \mathbb{R}_+^2\to\mathbb{R}_+$ is positive semi-definite.
$(s,t)\to s\wedge t$

$\square$

#15: WTS: $e^{-|t-s|}$ is positive semi-definite

Proof: define $(f_t)_{t\in\mathbb{R}_+}$ by $f_t(x) := \begin{cases} \sqrt{2}e^{-x+t}, & x \geq t \\ 0, & x < t \end{cases}$

Then, assume WOLG $t > s$, we have:

$$\langle f_t, f_s \rangle_{L^2} = \int_{\mathbb{R}} f_t(x) f_s(x)\, dx = \int_t^\infty f_t(x) f_s(x)\, dx$$

change variable: $y = x + s - t \Rightarrow$

$$= \int_s^\infty f_s(y) f_s(y+t-s)\, dy$$

$$= \int_s^\infty 2e^{-2y+3s-t}\, dy = e^{3s-t} e^{-2s}$$

$$= e^{s-t} = e^{-|t-s|}$$

Now, pick $n \in \mathbb{N}$ arbitrary, $t_1 \cdots t_n \subset \mathbb{R}_+$, $a \in \mathbb{R}^n$

we have $(C)_{ij} := e^{-|t_i - t_j|}$, $i,j \in \{1, \cdots n\}$

$$a^T C a = \sum_{i,j=1}^n a_i a_j C_{ij} = \sum_{i,j=1}^n a_i a_j \langle f_{t_i}, f_{t_j} \rangle_{L^2}$$

$$= \left\langle \sum_{i=1}^n a_i f_{t_i}, \sum_{j=1}^n a_j f_{t_j} \right\rangle_{L^2}$$

$$= \left\| \sum_{i=1}^n a_i f_{t_i} \right\|_{L^2} \geq 0$$

Since our choices of n, $t_1 \cdots t_n$, and a are arbitrary, we're done.

$\square$

#16. Given $X \sim \mathcal{N}(a, C) \in (\mathbb{R}^{\infty})^d$, C is non-singular.

WTS: ① $F_X(x)$ is absolutely continuous and

② $P_X(x) = \dfrac{1}{(\det C)^{\frac{1}{2}} (2\pi)^{d/2}} e^{-\frac{1}{2}\langle C^{-1}(x-a), (x-a)\rangle}$

Proof: Let $N := (N_1, \cdots, N_d)^T$, where $N_i \sim \mathcal{N}(0,1)$ iid, $\forall i$.

$\Rightarrow P_N(n) = \prod_{i=1}^{d} \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}n_i^2} = \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}n^T n}$, where $n^T n = \langle n, n\rangle$

Now, C is positive-definite (non-singular)

$\Rightarrow C = O^T D O$, where $O^T O = I$, $D := \mathrm{diag}(\lambda_1, \cdots, \lambda_d)$.

$= (\sqrt{D}\,O)^T(\sqrt{D}\,O)$

Let $\hat{X} := X - a$, $\hat{X} \sim \mathcal{N}(0, C)$

But $\begin{cases} E\,(\sqrt{D}\,O)^T N = 0 \\ E\,(\sqrt{D}\,O)^T N\,((\sqrt{D}\,O)^T N)^T = O^T\sqrt{D}\,E N N^T \sqrt{D}\,O = C \end{cases} \Rightarrow \hat{X} = (\sqrt{D}\,O)^T N$

① $\Rightarrow |dF_X(x)| = P_X(x)|dx| \leq |dx| = |d\hat{x}| \leq \det(\sqrt{D}\,O)^T |dn|$

Hence $F_X(x)$ is absolutely continuous w.r.t. Lebesgue measure.

②. Since C is non-singular, $\frac{1}{\sqrt{D}} := \mathrm{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \cdots, \frac{1}{\sqrt{\lambda_d}}\right)$ is well-defined.

$\Rightarrow N = [(\sqrt{D}\,O)^T]^{-1}\hat{X} = (O^T\sqrt{D})^{-1}\hat{X} = \frac{1}{\sqrt{D}} O\,\hat{X}$

Since $P_X(x)|dx| = P_N(n)|dn|$, we have:

$$P_X(x) = \frac{|dn|}{|dx|} P_N\left(\frac{1}{\sqrt{D}} O\,\hat{x}\right) = |\sqrt{D}\,O|^{-1} \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}\left(\frac{1}{\sqrt{D}}O\hat{x}\right)^T\left(\frac{1}{\sqrt{D}}O\hat{x}\right)}$$

$$= \frac{1}{|D|^{1/2}(2\pi)^{d/2}} e^{-\frac{1}{2}\hat{x}^T C^{-1}\hat{x}} = \frac{1}{|C|^{1/2}(2\pi)^{d/2}} e^{-\frac{1}{2}\langle C^{-1}(x-a), (x-a)\rangle}$$

□

#17. ①. $E|X_s - X_t|^2 = E X_s + E X_t - 2E X_s X_t$

$$= r(s,s) + r(t,t) - 2r(s,t) \leq |s-t|^2 \quad (*)$$

If $(X)_t$ is Gaussian w/ $a(t) \equiv 0$, $r(s,t)$ satisfying $(*)$, then by Kolmogorov-Chentsov, $\exists$ a continuous $(\tilde{X})_t$ and $\forall t \in \pi$, $P(\tilde{X}_t \neq X_t) = 0$.

Now, for $\forall n \in \mathbb{N}$, $\forall (t_1 \cdots t_n) \subset \pi$, we have

$$X_{t_1 \cdots t_n} \sim \mathcal{N}(0, (r(t_i, t_j))_{i,j \in \{1,\cdots,n\}})$$

Let $N_{t_1 \cdots t_n} := \bigcup_{i=1}^{n} \{\tilde{X}_{t_i} \neq X_{t_i}\} \Rightarrow P(N_{t_1 \cdots t_n}) = 0$.

$\Rightarrow \tilde{X}_{t_1 \cdots t_n} = X_{t_1 \cdots t_n}$ P-a.s. $\Rightarrow \tilde{X}_{t_1 \cdots t_n} \stackrel{d}{=} X_{t_1 \cdots t_n}$

$\Rightarrow \tilde{X}_{t_1 \cdots t_n} \sim \mathcal{N}(0, (r(t_i, t_j))_{i,j \in \{1,\cdots,n\}})$

Since it's true for $\forall n$, $\forall (t_1 \cdots t_n) \subset \pi$,

$\tilde{X}$ is Gaussian and continuous.

$\square$

#17. ②. Fernique's sufficient condition for continuity of paths.

Let $(X)_{t\in\pi}$, $\pi \subset \mathbb{R}_+$, be Gaussian w/ $\begin{cases} a(t) = E(X_t) \\ r(s,t) = Cov(X_s, X_t) \end{cases}$

satisfying: $a(t) \equiv 0$; $r(s,t) = r(s+\tau, t+\tau)$, $\forall \tau, t \begin{cases} s+\tau \in \pi \\ t+\tau \in \pi \end{cases}$

strictly stationary.

Let $R(t) := \max\{\sqrt{r(s,s)} : 0 \le s \le t\}$, $t \in [0,1]$.

w/ $\int_0^1 \frac{R(t)}{t\sqrt{\log t^{-1}}}\, dt < \infty$

(Then ~~$X_\cdot(\omega) \in C^{(0)}(\pi)$ for $\forall \omega$~~.

Nice,
but: there is a continuous modification.

One can easily cook a discontinuous modification of any continuous process, leaving $a(t)$ and $r(s,t)$ intact

#18. Given $(X_0, X_1, \cdots, X_n)^T$ a Gaussian vector

WTS: $E(X_0 | X_1, \cdots, X_n) = c_0 + \sum_{i=1}^{n} c_i X_i$ for some $(c_0, \cdots, c_n)$.

Proof: Let $Y := (X_1, \cdots, X_n)^T$

EYY^T is positive semi-definite $\Rightarrow$ $EYY^T = O^* D_Y O$

when $D_Y = \text{diag}(\lambda_1, \cdots, \lambda_n)$

If $\exists\, i \in \{1, \cdots, n\}$ s.t. $\lambda_i = 0$, we delete X_i from Y

$\Rightarrow$ Obtaining $\hat{Y}$ s.t. $0 \notin \text{spectrum}(E\hat{Y}\hat{Y}^T)$.

Note: $\lambda_i = 0 \Rightarrow \exists\, V \in E_{\lambda_i}(EYY^T)$ s.t. $E(V^T Y)^2 = E V^T Y Y^T V = 0$

$\Rightarrow X_i \in \text{span}\{X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n\}$.

Now, let $\hat{X} := EX_0 + C_{X_0\hat{Y}} C_{\hat{Y}}^{-1} (\hat{Y} - E\hat{Y})$

where $\begin{cases} C_{X_0\hat{Y}} := E(X_0 - EX_0)(\hat{Y} - E\hat{Y})^T \\ C_{\hat{Y}}^{-1} := (E(\hat{Y} - E\hat{Y})(\hat{Y} - E\hat{Y})^T)^{-1} \end{cases}$

$\Rightarrow E(\hat{X} - EX_0)(\hat{Y} - E\hat{Y})^T = E\, C_{X_0\hat{Y}} C_{\hat{Y}}^{-1} (\hat{Y} - E\hat{Y})(\hat{Y} - E\hat{Y})^T$

$= C_{X_0\hat{Y}} C_{\hat{Y}}^{-1} C_{\hat{Y}} = E(X_0 - EX_0)(\hat{Y} - E\hat{Y})^T$

$\Rightarrow E(X_0 - \hat{X})(\hat{Y} - E\hat{Y})^T = 0 \Rightarrow X_0 - \hat{X}$ & $\hat{Y}$ are uncorrelated.

$\text{span}(\hat{Y}^T) = \text{span}(Y^T) \Rightarrow X_0 - \hat{X}$ & Y are uncorrelated.

Since $(X_0 - \hat{X}, Y^T)$ are jointly Gaussian $\Rightarrow X_0 - \hat{X}$ & Y are independent.

$\Rightarrow E(X_0 - \hat{X}) f(X_1 \cdots X_n) = 0$, $\forall f$

$\Rightarrow E(X_0 | X_1 \cdots X_n) = \hat{X} = EX_0 + C_{X_0\hat{Y}} C_{\hat{Y}}^{-1} (\hat{Y} - E\hat{Y})$

$= c_0 + \sum_{i=1}^{n} c_i X_i$, $c_i = 0$ for $X_i \in \{Y \setminus \hat{Y}\}$ $\square$

#19. ⓒ. WTS: standard Ornstein-Uhlenbeck has a continuous modification

Proof: X be standard Ornstein-Uhlenbeck

$$E|X_t - X_s|^4 = 3\left(E|X_t - X_s|^2\right)^2 \quad \text{since } X_t - X_s \sim N(0, e^{-|t-s|})$$

$$= 3\left[E\left(X_t^2 - 2X_tX_s + X_s^2\right)\right]^2$$

$$= 3\left[2\left(1 - e^{-|t-s|}\right)\right]^2$$

$$\leq 12\,|t-s|^2$$

Now, apply Kolmogorov-Chentsov w/ $\alpha = 4$, $\beta = 1$, $C = 12$.

$\Rightarrow$ $\exists$ continuous modification for standard Ornstein-Uhlenbeck

$\square$

#19. ⓑ. WTS. $E(X_4 \mid X_1, X_2, X_3)$

Since X is Gaussian w/ mean 0, we have from #18 that

$E(X_4 \mid X_1, X_2, X_3) = 0 + C_{X_4\hat{X}} C_{\hat{X}}^{-1} \hat{X}^T$, where $\hat{X} := (X_1, X_2, X_3)$.

$$C_{X_4\hat{X}} = (C(X_4, X_1), C(X_4, X_2), C(X_4, X_3))$$
$$= (e^{-3}, e^{-2}, e^{-1})$$

$$C_{\hat{X}} = \left(e^{-|i-j|}\right)_{i,j\in\{1,2,3\}} = \begin{pmatrix} 1 & e^{-1} & e^{-2} \\ e^{-1} & 1 & e^{-1} \\ e^{-2} & e^{-1} & 1 \end{pmatrix}$$

By Cramer's rule, we find:

$$C_{\hat{X}}^{-1} = \frac{1}{1-2e^{-2}+e^{-4}} \begin{pmatrix} 1-e^{-2} & e^{-3}-e^{-1} & 0 \\ e^{-3}-e^{-1} & 1-e^{-4} & e^{-3}-e^{-1} \\ 0 & e^{-3}-e^{-1} & 1-e^{-2} \end{pmatrix}$$

$$\Rightarrow E(X_4 \mid X_1, X_2, X_3) = (e^{-3}, e^{-2}, e^{-1})\, C_{\hat{X}}^{-1} \hat{X}^T$$

$$= \frac{1}{1-2e^{-2}+e^{-4}} \Big(X_1(e^{-3}-e^{-5}+e^{-5}-e^{-3}) + X_2(e^{-6}-e^{-4}+e^{-2}-e^{-6}+e^{-4}-e^{-2})$$
$$+ X_3(e^{-5}-e^{-3}+e^{-1}-e^{-3}) \Big)$$

$$= \frac{e^{-1}-2e^{-3}+e^{-5}}{1-2e^{-2}+e^{-4}} X_3$$

#20. WTS: $\forall\, (X_t)_{t\in\mathbb{T}}$ centered Gaussian process s.t. for $t>s$ $X_t - X_s$ is independent of $\sigma(X_r, r\le s)$, $\exists$ a nondecreasing $f: \mathbb{R}_+ \to \mathbb{R}_+$ s.t. X has the same f.d.d. as $Y := W\circ f$

Proof:

$(X_t)_{t\in\mathbb{T}}$ centered Gaussian process $\Rightarrow \begin{cases} \exists\, a(t) := E X_t, & \forall\, t\in\mathbb{T} \\ \exists\, C(t,s) := Cov(X_t, X_s), & \forall\, t,s \end{cases}$

$\forall\, n\in\mathbb{N}$, $\forall\, (t_1,\dots,t_n)\subset\mathbb{T}$, we have:

$X_{t_1\dots t_n} \sim \mathcal{N}(0, K_{t_1\dots t_n})$, $K_{t_1\dots t_n} := E X_{t_1\dots t_n} X_{t_1\dots t_n}^T$

$X_t - X_s \perp \sigma(X_r, r\le s) \Rightarrow K_{t_1\dots t_n} = \left(Cov(X_{t_i}, X_{t_j})\right)_{i,j\in\{1,\dots,n\}}$

$= \left(Var\, X_{t_i\wedge t_j}\right)_{i,j\in\{1,\dots,n\}}$.

Now, define $f(t) := C(t,t)$, since $Y = W\circ f$, we have:

$Y_{t_1\dots t_n} \sim \mathcal{N}(0, \widetilde{K}_{t_1\dots t_n})$, $\widetilde{K}_{t_1\dots t_n} := E Y_{t_1\dots t_n} Y_{t_1\dots t_n}^T$

Since W is Wiener $\Rightarrow E Y_{t_1\dots t_n} Y_{t_1\dots t_n}^T = \left(f(t_i)\wedge f(t_j)\right)_{i,j\in\{1,\dots,n\}}$.

But if $t>s$, $f(t) = C(t,t) = Var\, X_t = Var\, X_s + Var\, X_{t-s} \ge Var\, X_s = f(s)$

$\Rightarrow f$ is non-decreasing $\Rightarrow f(t_i)\wedge f(t_j) = f(t_i\wedge t_j) = C(t_i\wedge t_j, t_i\wedge t_j)$

$\Rightarrow K_{t_1\dots t_n} = \left(Var\, X_{t_i\wedge t_j}\right)_{ij} = \left(C(t_i\wedge t_j, t_i\wedge t_j)\right)_{ij} = \widetilde{K}_{t_1\dots t_n}$

$\Rightarrow X_{t_1\dots t_n} \overset{d}{=} Y_{t_1\dots t_n}$

Since, n, $(t_1\dots t_n)$ are arbitrarily chosen, we're done. $\square$

#21. (b). WTS. $\exists$ P.P.P w/ leading measure μ on $\mathbb{R}^d$ (σ-finite).

Proof: Since μ is σ-finite on $\mathbb{R}^d$, $\exists \{U_i\}_{i\in\mathbb{N}}$ s.t.

$\bigcup_{i\in\mathbb{N}} U_i = \mathbb{R}^d$, $U_i \cap U_j = \phi$, $i \neq j$, $\mu(U_i) < \infty$ $\forall i \in \mathbb{N}$.

①. For chosen $i \in \mathbb{N}$, take $(N_k^i)_{k\in\mathbb{N}}$ i.i.d. s.t. $P(N_k^i \in A) = \dfrac{\mu(A\cap U_i)}{\mu(U_i)}$

②. Let $M^i \sim \text{Poisson}(\mu(U_i))$, and independent of $(N_k^i)_{k\in\mathbb{N}}$

$\Rightarrow$ $P(M^i > t) = e^{-t\mu(U_i)}$,

③. Define $(X_A^i)_{A\in\mathcal{B}(\mathbb{R}^d)}$ s.t. $X_A^i := \sum_{k=1}^{M^i} \mathbb{1}_{\{N_k^i \in A\}}$

$\Rightarrow$ $(X_A^i)_{A\in\mathcal{B}(\mathbb{R}^d)}$ is P.P.P w/ leading measure $\mu|_{U_i}$

Now, define $(X)_{A\in\mathcal{B}(\mathbb{R}^d)}$ by $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $X_A := \sum_{i=1}^{\infty} X_A^i$

$\Rightarrow$ (X_A) is P.P.P w/ leading measure μ.

by the superposition of Poisson process and the fact that U_i's are disjoint and $\mu = \sum_{i\in\mathbb{N}} \mu|_{U_i}$.

$\square$

#21. ⓑ. WTS: $\exists$ white noise w/ leading measure μ on $\mathbb{R}^d$

Define $W_i(A) \sim \mathcal{N}(0, \mu(A \cap U_i))$, $\mu(U_i) < \infty$.

$$\mathrm{Cov}(W_i(A), W_i(B)) = \mu(A \cap B \cap U_i)$$

$\Rightarrow$ Since $\mathrm{Cov}(W_i(A), W_i(B))$ is positive semi-definite,

$$\text{Let } (W_i)_{A \in \mathcal{B}\mathbb{R}^d} \text{ w/ } \begin{cases} E(W_{iA}) \equiv 0 \\ C_i(A,B) := \mu|_{U_i}(A \cap B). \end{cases}$$

$\forall n$, $\forall A_1 \cdots A_n \subset \mathcal{B}(\mathbb{R}^d)$, $a_1 \cdots a_n \in \mathbb{R}$, we have.

$$\sum_{i,j} a_i a_j C_i(A_i A_j) = \left\| \sum_{i=1}^{n} a_i I_{A_i} \right\|_{L^2(\mu)} \geq 0.$$

$$\Rightarrow (W_i)_{A_1 \cdots A_n} \sim \mathcal{N}\left(0, (C_i(A_i, A_j))_{i,j \in \{1 \cdots n\}}\right)$$

$$\Rightarrow (W_i)_{A \in \mathcal{B}(\mathbb{R}^d)} \text{ is Gaussian}$$

Now, since μ is σ-finite, $\exists \{U\}_{i \in \mathbb{N}}$ s.t. $\bigcup_{i=1}^{\infty} U_i = \mathbb{R}^d$ s.t. $U_i \cap U_j = \phi$, $i \neq j$, and $\mu(U_i) < \infty$, $\forall i$.

$\Rightarrow$ $(W_i)_A$ is Gaussian w/ leading measure $\mu|_{U_i}$

$$\Rightarrow W := \sum_{i=1}^{\infty} W_i, \quad EW = \sum_{i=1}^{\infty} 0 = 0$$

$$C(A,B) = \sum_{i=1}^{\infty} \mu|_{U_i}(A \cap B) = \mu(A \cap B)$$

Therefore $(W)_{A \in \mathcal{B}(\mathbb{R}^d)}$ is Gaussian w/ leading measure μ.

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$\square$